Asymptotic resonance, modes interaction and subharmonic bifurcation*

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1 Introduction

The principal purpose of this work is to provide a theoretical explanation of certain bifurcation phenomena that appear in some classes of Lagrangian systems. As a consequence, our main results will propose a new method for the detection of secondary bifurcations that relies on very simple and explicitly computable conditions.

Our interest in these topics was originally motivated by the paper [1], where the authors look for periodic solutions in the N–particle Fermi–Pasta–Ulam β–model ([7]). The dynamics is governed by the equation

$$\ddot{x} + Ax + W'(x) = 0, \tag{1}$$

where

$$Ax \cdot x = \sum_{i=0}^{N} (x_{i+1} - x_i)^2, \qquad W(x) = \frac{1}{4} \sum_{i=0}^{N} (x_{i+1} - x_i)^4$$

and $x = (x_1, \ldots, x_N) \in \mathbf{R}^N$ denotes the displacements of the particles from equilibrium. In [1] the spatially periodic case is considered, but essentially the

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same results hold for the simpler fixed ends problem, where $x_0 = x_{N+1} = 0$, which we use as a model throughout this paper.
After the time scaling

$$x(t) = u(\omega t), \qquad \text{with} \qquad \omega = \frac{2\pi}{T},$$

the search for T–periodic solutions to (1) is formulated by the boundary value problem

$$\begin{cases} \omega^2 \ddot{u} + Au + W'(u) = 0 \\ u \quad 2\pi - periodic \end{cases} \tag{P_ω}$$

the frequency ω being now a free (bifurcation) parameter.
By means of computer–assisted arguments and numerical methods, in [1] the authors produce the bifurcation diagram reported in Figure 1.

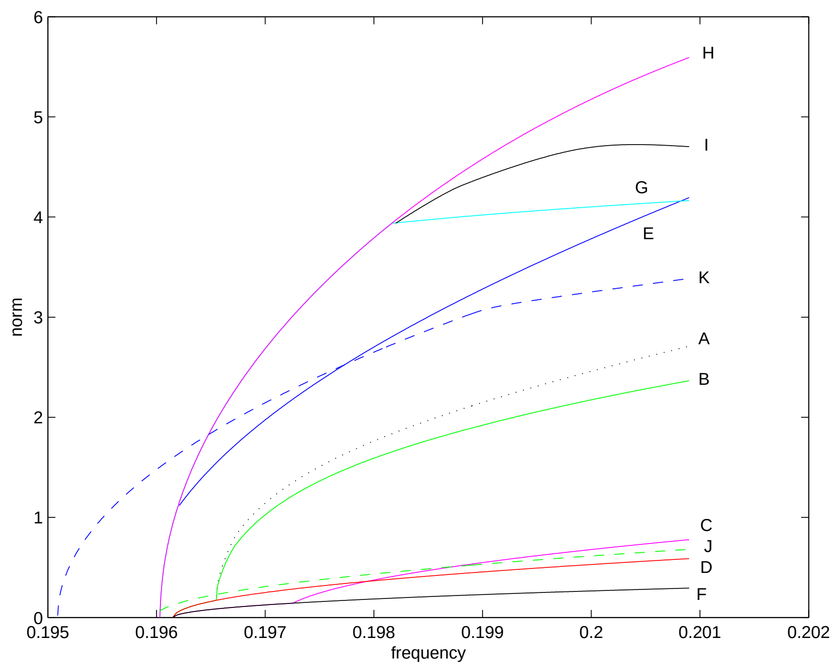

Figure 1: The bifurcation diagram of [1]

Since the N positive eigenvalues μ_j^2 of A are explicitly known ([18], for instance) and satisfy the fundamental property

$$(Q) \qquad \frac{\mu_i}{\mu_j} \notin \mathbf{Q} \qquad \text{for all } i \neq j,$$

([4], or Section 8), it is a standard fact from bifurcation theory ([6], [2]) that every frequency $\omega_{jk} = \mu_j / k$, $j = 1, \ldots, N$, $k \in \mathbf{N}^+$, is a bifurcation point for the system from the trivial line $(\omega, 0)$.

The local primary branches that depart from these points are small deformations of the normal modes of oscillation of the linearization at zero, and carry solutions with bounded minimal periods. Such is the case, in Figure 1, for the branches labelled by F, H, J and K, which in [1] are then numerically continued until other branches bifurcate from them.

The analysis of Figure 1 and of the results in [1] poses a number of problems from the theoretical point of view.

First of all, the authors of [1] point out that the approach they follow to construct the diagram in the figure is driven by a *conjecture*: the appearance of secondary bifurcation branches is related to *anomalies* in the distribution of the frequencies ω_{jk} on the real line. In the picture, for example, the bifurcation frequencies of the primary branches F, H, J and K appear as a tight cluster among all other frequencies, and those of H and J as an even tighter subcluster.

A further analysis of the harmonic energy along the secondary branches shows that these, in contrast to the primary ones, carry solutions that "mix modes"; in this context, another role of the frequencies involved in the cluster is that they seem to decide which modes are mixed along a given secondary branch. The mixing however turns out to be quite asymmetric and the analysis of [1] leads to believe that *prevalently* a primary branch in a cluster interacts and mix modes with other branches located close to it and *at its left*.

These considerations are the result of mainly numerical computations and call for a theoretical understanding. This is the aim of the present paper, for a class of systems including the Fermi–Pasta–Ulam model. The main questions we address are the mechanism of formation of secondary bifurcations, the reasons for the asymmetry in the interactions, and the discussion of the notions we will introduce to prove the main results.

Of course we will test the outcome on the model but, we believe, the instrument we construct is flexible enough to be fitted to more general problems.

Our results are formulated for problem (1) where now the $N \times N$ matrix A and the potential $W : \mathbf{R}^N \to \mathbf{R}$ satisfy the assumptions

(A) *A is symmetric, positive definite and has only simple eigenvalues whose square roots are pairwise independent over $\mathbf{Q}$;*

(W) *W is an homogeneous polynomial of degree four and $W > 0$ outside zero.*

We denote by

$$\mu_1 < \ldots < \mu_N$$

the square roots of the eigenvalues of A and we call *characteristic values* the numbers (frequencies)

$$\omega_{jk} = \frac{\mu_j}{k}, \qquad j = 1, \ldots, N, \quad k \in \mathbf{N}^+.$$

The same arguments as for the Fermi–Pasta–Ulam problem show that from every $(\omega_{jk}, 0)$ there departs a *primary* branch Γ_{jk} of solutions to (P_ω).
Our first concern is to establish sufficient conditions for the existence of bifurcation points on the branches Γ_{jk}, or, in other words, of *secondary bifurcations*.
We enter here a word of caution. In contrast to other possible approaches, we do not construct a global extension of Γ_{jk}, and, even worse, we will be forced to work as close as possible to the base point $(\omega_{jk}, 0)$; close to this point uniqueness arguments prevent the existence of secondary bifurcations, so that a delicate construction must be carried out to obtain a sort of "local but not too local" result.
We restrict our attention to a pair of (close) characteristic values

$$\omega_{ih} < \omega_{jk} \tag{2}$$

with $i \neq j$, and to the associated primary branches Γ_{ih} and Γ_{jk}. Our main results give a sufficient condition for a secondary bifurcation to appear on the right branch Γ_{jk}, or on the left branch Γ_{ih} respectively, because of their mutual interaction. We denote by ω^R and ω^L the nearest characteristic values respectively to the right and to the left of ω.

Theorem 1.1 (Bifurcation from the right branch) *Under the assumptions* (A) *and* (W), *suppose moreover that*

$$6\mu_i^2 W(e_j) - \mu_j^2 W''(e_j) \cdot e_i^2 < 0 \tag{3}$$

for some for some $i \neq j$. *Then there exists* $\sigma > 0$ *such that, if for some* $h, k \in \mathbf{N}^+$ *there results*

$$\omega_{ih} < \omega_{jk} \qquad \text{and} \qquad \frac{\omega_{jk} - \omega_{ih}}{\omega_{jk}^R - \omega_{jk}} < \sigma, \tag{4}$$

then the primary branch Γ_{jk} *supports a secondary bifurcation in the interval* $(\omega_{jk}, \omega_{jk}^R)$.

Theorem 1.2 (Bifurcation from the left branch) *Under the assumptions* (A) *and* (W), *suppose moreover that*

$$6\mu_j^2 W(e_i) - \mu_i^2 W''(e_i) \cdot e_j^2 > 0 \tag{5}$$

for some for some $i \neq j$. *Then there exists* $\sigma > 0$ *such that, if for some* $h, k \in \mathbf{N}^+$ *there results*

$$\omega_{ih} < \omega_{jk} \qquad \text{and} \qquad \frac{\omega_{jk} - \omega_{ih}}{\omega_{ih} - \omega_{ih}^L} < \sigma, \tag{6}$$

then the primary branch Γ_{ih} *supports a secondary bifurcation in the interval* $(\omega_{jk}, 2\,\omega_{ih} - \omega_{ih}^L)$.

Notice that the two results are very similar, but not completely symmetric, and indeed they describe two different situations.
The presence of secondary bifurcations will be detected via a result by Kielhöfer, ([10]), implying that along an analytic branch of solutions, every point of discontinuity for the Morse index of the solutions (seen as critical points of the usual action functional) is either a bifurcation or a turning point; the last possibility will be ruled out by trivial local arguments.
In the statement of the theorems we do not provide any estimate for the quantity σ. The practical way to check the assumption is thus to locate cases in which the fraction in (4) or (6) are not bounded away from zero along some divergent sequences. This concept plays a fundamental role in our research and motivates the following definition.

Definition 1.3 *We say that μ_i is* right asymptotically resonant *with μ_j, if there exist diverging sequences $h_n, k_n \in \mathbf{N}$ such that*

$$\omega_{ih_n} < \omega_{jk_n} \quad \forall n \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{\omega_{jk_n} - \omega_{ih_n}}{\omega^R_{jk_n} - \omega_{jk_n}} = 0. \tag{7}$$

Left asymptotical resonance *is defined similarly to restate (6) (see Section 5).*

The notions of right and left asymptotic resonance make precise the informal idea of clustering appearing in [1], for an interaction involving a pair of primary branches. Essentially, they are closeness assumptions on the two frequencies ω_{jk_n} and ω_{ih_n}, whose relative distance has to be much smaller than their distance to the remaining characteristic values (on the right or left, respectively). The concept depends only on the arithmetical properties of the numbers μ_j, and is therefore related to the linear part of the problem.
The other ingredient of Theorems 1.1 and 1.2 are the *nonlinear coupling conditions* (3) and (5). These conditions fully account for the asymmetry in the interactions already mentioned. Indeed defining

$$W_{ij} = 6\mu_i^2 W(e_j) - \mu_j^2 W''(e_j) \cdot e_i^2, \qquad i,\, j = 1, \ldots, N,$$

the theorems state that in presence of the correct asymptotic resonance a secondary bifurcation appears on the *right* branch Γ_{jk} if $W_{ij} < 0$, or on the *left* branch Γ_{ih} if $W_{ji} > 0$.

It is thus the simultaneous occurrence of the appropriate nonlinear coupling with the appropriate asymptotic resonance that produces a secondary bifurcation; we believe that the understanding of this phenomenon is the core of our work.

As a corollary of the previous theorems, we obtain the following Birkhoff–Lewis type result, which also accounts for the subharmonic character of the secondary bifurcations.

Corollary 1.4 *Under the assumptions* (A) *and* (W)*, suppose moreover that*

$$W_{ij} = 6\mu_j^2 W(e_i) - \mu_i^2 W''(e_i) \cdot e_j^2 \neq 0 \tag{8}$$

for some for some $i \neq j$*. If, according to* $W_{ij} < 0$ *or* $W_{ij} > 0$*, either* μ_i *is right asymptotically resonant with* μ_j*, or* μ_j *is left asymptotically resonant with* μ_i*, then there exists a sequence of periodic solutions to (1), whose* $\mathcal{C}^2$ *norms tend to zero and whose minimal periods tend to infinity.*

In the Fermi–Pasta–Ulam problem our results will show that this Birkhoff–Lewis type theorem holds for every dimension N. The same conclusion could be obtained as a byproduct of a result in [15], by showing that the fourth order Birkhoff normal form is nonresonant and nondegenerate (in the sense of Kolmogorov). We point out that this approach may fail in the more general framework identified by assumptions (A) and (W) because of two reasons. First there might be fourth order resonances among the μ_j's, preventing the normal form to be nonresonant; this fact seems to be unrelated to the notion of asymptotic resonance. Moreover, even in the absence of resonances, the Kolmogorov nondegeneracy condition needs not be fulfilled; although this condition holds generically, it is very hard to check, for one should know the explicit expression of the normal form, which rarely happens. This has to be compared with the simple and computable form of the nonlinear coupling conditions (3) and (5).

The above results are contained in Sections 2–6. The remaining part of the paper is devoted to the analysis of conditions used in the main theorems and to the application of this analysis to the Fermi–Pasta–Ulam problem (for recent results on this problem see [15], and [16] together with its extensive bibliography). Here we use some number theoretical arguments which may by unfamiliar to the analyst.

In particular, a complete study of the notion of asymptotic resonance shows that it is equivalent to a set of Diophantine equations depending on the coefficients of the *ternary relations* (null linear combinations with integer coefficients involving exactly three μ_j's) that may be present.

As a conclusion, μ_i is proved to be asymptotically resonant with any other μ_j, whenever the ternary relations involving them, if any exist, are of a special type (see Section 7).

The abstract Diophantine equations are then fully solved for the Fermi–Pasta–Ulam model, obtaining the complete list of asymptotic resonances for every value of N. As a particular case we obtain that μ_i is asymptotically resonant with any other μ_j as soon as $N+1$ is not a multiple of 3.

Finally we analyze the validity of the nonlinear coupling conditions (3) and (5) for the Fermi–Pasta–Ulam model. We obtain, for every N, that

$$W_{ij} \quad \text{is} \quad \begin{cases} > 0 & \text{if } i+j = N+1 \\ < 0 & \text{otherwise.} \end{cases}$$

The relative preponderance of negative terms explains, we believe, numerical results such as those of [1].
The plan of the paper is as follows. In Section 2 we establish the abstract setting and we construct the primary bifurcation branches with their asymptotic expansions and properties. In Section 3 we begin the study of the Morse index along the primary branches. Section 4 introduces the main asymptotic arguments and establishes that a variation of the Morse index takes place along a right branch under certain conditions. In Section 5 the same is done for the left branches. The proofs of the main results are given in Section 6. The concept of asymptotic resonance is studied in the abstract case in Section 7, where necessary and sufficient conditions for its validity are established. The Diophantine equations for asymptotic resonance are completely solved in Section 8 for the Fermi–Pasta–Ulam problem and finally, in Section 9, for the same problem, the nonlinear coupling conditions are established.

Notation. We denote by $\lfloor a \rfloor$ and $\{a\}$ the integer and fractional parts respectively of a real number a. If $m, n \in \mathbf{N}$, the symbol (m, n) stands for the greatest common divisor of m and n, while $m \mid n$ means that m divides n. Finally $x_n \asymp y_n$ means that $ax_n \leq y_n \leq bx_n$ definitely holds for suitable positive constants a, b.

2 Primary bifurcations and Lyapunov orbits

Solutions of (P_ω) are critical points of the *analytic* action functional

$$J(\omega, u) = J_\omega(u) = \frac{\omega^2}{2}\int_0^{2\pi} |\dot{u}|^2\,dt - \frac{1}{2}\int_0^{2\pi} Au\cdot u\,dt - \int_0^{2\pi} W(u)\,dt \tag{9}$$

on the Hilbert space $H^1_{2\pi} = \{u \in H^1_{loc}((0, 2\pi); \mathbf{R}^N) \mid u(t + 2\pi) = u(t)\ \forall t\,\}$.
In view of the development of our arguments, and in particular in order to gain a uniqueness property for the (primary) bifurcation branches, we restrict the functional J_ω to the space

$$H = \{u \in H^1_{2\pi} \mid u \text{ is even}\,\}.$$

The space H is a *natural constraint* for the functional J_ω, in the sense that critical points of J_ω constrained to H are free critical points of J_ω considered as a functional on the whole space $H^1_{2\pi}$, and hence solutions of problem (P_ω). These statements derive from the fact that problem (P_ω) is autonomous and are easy to check.
The principal scope of this section is to locate and describe all the bifurcation branches from the trivial critical line $(\omega, 0)$, $\omega \in \mathbf{R}^+$.

It is well known that J'_ω is a Fredholm map of index zero. Denoting by

$$L_\omega v = -\omega^2 \ddot{v} - Av$$

the derivative of J_ω at $u = 0$, we have that up to the Riesz isomorphism, L_ω is a symmetric operator and we can write $H = \text{Ker } L_\omega \oplus \text{Im } L_\omega$. A standard analysis shows that L_ω is not invertible precisely for

$$\omega = \omega_{jk} = \frac{\mu_j}{k}, \quad j = 1, \ldots, N, \quad k \in \mathbf{N}^+.$$

In this case, moreover, condition (Q) implies that Ker $L_{\omega_{jk}}$ is one–dimensional and is spanned by

$$\varphi_{jk}(t) = \frac{1}{\sqrt{\pi}} \cos(kt) e_j.$$

The functions φ_{j1} are usually called *normal modes* of oscillation, while the φ_{jk}'s with $k > 1$ are the associated higher harmonics; the choice we made of the coefficients makes all these functions normalized in L^2.
Notice that, for instance by a result of Stuart ([17]), all the elements of the set

$$\Omega = \bigcup_{j=1}^{N} \Omega_j, \qquad \text{where} \qquad \Omega_j = \{\frac{\mu_j}{k} \mid k \in \mathbf{N}^+\},$$

are bifurcation points. However, we are not granted generally that there is a *branch* originating at each point of Ω. The most direct way to obtain a branch is the application of the Crandall–Rabinowitz bifurcation theorem (see [6] or [2]). The required transversality condition is immediate to check, so that we obtain the following classical statement.

Proposition 2.1 *(Primary bifurcations). Assume that* (A) *and* (W) *hold. Then, for every* $\omega_{jk} \in \Omega$*, the set of nontrivial critical points of* J_ω *around* $(\omega_{jk}, 0)$ *is a unique analytic curve* $\xi \mapsto (\omega_\xi, u_\xi)$ *whose expansion near* $\xi = 0$ *satisfies*

$$\begin{cases} \omega_\xi = \omega_{jk} + \dfrac{3W(e_j)}{2\pi\mu_j k}\, \xi^2 + O(\xi^3) & \text{in } \mathbf{R} \\ u_\xi = \xi \varphi_{jk} + O(\xi^2) & \text{in } H. \end{cases} \tag{10}$$

Notice that since $W > 0$ outside zero, every ω_{jk} is a supercritical bifurcation point. The previous result is just a particular case of theorems that can be found, for instance, in [2] and [11]; the analyticity of the branch is explicitly considered in [3]. These branches (especially when $k = 1$) are sometimes called *Lyapunov orbits* because they can also be constructed by the Lyapunov Center Theorem; also in this case though, property (Q) is essential.

¿From now on, we are interested in bifurcations from the primary branches, or, in other words, in *secondary* bifurcations. However these are clearly impossible to find unless the primary branches are extended outside the uniqueness regions defined by Proposition 2.1. As we will see in a while, the way to carry out these extensions is first to focus on the *principal* primary branches, namely the N branches corresponding to $k = 1$, and then to act on them with the map

$$\begin{cases} \omega \mapsto \omega/k \\ u(t) \mapsto u(kt), \end{cases} \tag{11}$$

which sends solutions of (P_ω) into solutions of $(P_{\omega/k})$. This map transforms the branch originating at $(\mu_j, 0)$ into a branch originating at $(\omega_{jk}, 0)$, which must coincide *locally* with the one already found in Proposition 2.1. The scaled branch however may differ from the other one for ω far from ω_{jk}, and this is where secondary bifurcations may take place.

In order to carry out this program, it is convenient to reparametrize the primary branches using the global variable ω instead of the local one ξ. This may be done locally, splitting each branch into two branches according to the sign of ξ. We do it first for the principal primary branches, pointing out at the same time some useful properties.

Looking at expansion (10) for $k = 1$, we see that we can choose small positive numbers r, R such that in

$$B_r(\mu_j) \times B_R(0) \subset \mathbf{R} \times H \tag{12}$$

the curve of nontrivial critical points enjoys the uniqueness property stated in the Proposition 2.1 and is *free of turning points*, apart from the trivial one $(\mu_j, 0)$. Reparametrizing the curve using ω instead of ξ, we obtain two branches

$$\omega \in [\mu_j, \mu_j + r) \;\mapsto\; \pm\, u_\omega \in B_R(0)$$

according to the sign of ξ; the symmetry of the two branches comes from the oddness of W and it is not relevant for our approach.
The function u_ω depends analytically on ω (except at $\omega = \mu_j$). Moreover, since choosing $\xi > 0$ in (10) and inverting we obtain

$$\xi = \left(\frac{2\pi\mu_j}{3W(e_j)} (\omega - \mu_j) \right)^{1/2} + O(\omega - \mu_j),$$

the expansion of u_ω as $\omega \to \mu_j^+$ is given by

$$u_\omega = (c_j(\omega - \mu_j))^{1/2} \varphi_{j1} + O(\omega - \mu_j) \tag{13}$$

in H, where we have set

$$c_j = \frac{2\pi\mu_j}{3W(e_j)}.$$

For future reference we also notice from (13) that in $B_r(\mu_j)\times B_R(0)$ every function lying on Γ_j has minimal period 2π. We are now ready to introduce the objects we will deal with from now on, namely convenient versions of the primary branches, restricted to a neighborhood where all the properties stated above hold true.

Definition 2.2 *For every $j = 1, \ldots, N$, we denote by Γ_j the graph of the function*

$$\omega \in [\mu_j, \mu_j + r) \ \mapsto \ u_\omega \in B_R(0)$$

and, for every integer k, by Γ_{jk} the image of Γ_j trough the map (11).

Of course each Γ_{jk} is a branch of solutions emanating from $(\omega_{jk}, 0)$; its domain is estimated simply, as a function of k, by

$$\omega_{jk} \le \omega < \omega_{jk} + \frac{r}{k}.$$

The branch Γ_{jk} shares the properties of Γ_j, among which the analyticity and the absence of turning points; these properties will all be used in Section 6.
The scaling procedure shows that

$$\Gamma_{jk} \subset B_{r/k}(\omega_{jk}) \times B_{kR}(0)$$

and moreover makes clear the dependence on k in the expansion of Γ_{jk}. This is reported in the following lemma.

Lemma 2.3 *For every $j = 1, \ldots, N$ and every $k \in \mathbf{N}^+$, the branch Γ_{jk} has the expansion*

$$u_\omega = (c_j k(\omega - \omega_{jk}))^{1/2}\varphi_{jk} + O(k(\omega - \omega_{jk})) \tag{14}$$

as $k(\omega - \omega_{jk}) \to 0^+$, where $O(k(\omega - \omega_{jk}))$ is in the L^∞ norm.

Proof. Expansion (13) says that on Γ_j

$$u(t) = (c_j(\mu - \mu_j))^{1/2}\frac{1}{\sqrt{\pi}}e_j \cos t + R(t; \mu - \mu_j),$$

where $||R(\cdot\,; \mu - \mu_j)|| \le C|\mu - \mu_j|$ as $\mu \to \mu_j^+$, for a suitable constant C independent of j. Evaluating this expansion at kt and setting $\omega = \mu/k$ we see that the principal part is the same as in (14), while the remainder takes the form $R(kt, k(\omega - \omega_{jk}))$. Since

$$\sup_t |R(kt, k(\omega - \omega_{jk}))| = \sup_t |R(t, k(\omega - \omega_{jk}))| \le C||R(\cdot, k(\omega - \omega_{jk}))||,$$

the statement follows. ■

The reason why we substitute H with L^∞ in the evaluation of the remainder term is that in this way we gain a property of uniformity: the remainder does not grow as $k \to \infty$. All the estimates of the next sections will be based on L^∞ type expansions.

3 Morse index estimates

We begin in this section the computation of the Morse index of the solutions lying on a given branch Γ_{jk}, namely we will look for subspaces of H where the quadratic form

$$J''_\omega(u_\omega) \cdot w^2 = \int_0^{2\pi} L_\omega w \cdot w \, dt - \int_0^{2\pi} W''(u_\omega) \cdot w^2 \, dt$$

is negative definite; here, as in the previous section, u_ω describes Γ_{jk} as ω varies. We set

$$E_{jk} = \text{span}\, \{\varphi_{ih} \mid \omega_{ih} > \omega_{jk}\} \subset H \quad \text{and} \quad \nu_{jk} = \dim E_{jk} < \infty$$

and we identify the subspace of constant functions with $\mathbf{R}^N$. We also recall that ω^L (resp. ω^R) is the largest (resp. smallest) element of Ω at the left (resp. right) of ω.
Denoting by $m(u_\omega)$ the Morse index of u_ω, the main result of this section is the following.

Theorem 3.1 *Under assumptions* (A), *and* (W), *for every* $j = 1, \ldots, N$ *and every* $k \in \mathbf{N}^+$, *the equality*

$$m(u_\omega) = \nu_{jk} + N + 1.$$

eventually holds as $\omega \to \omega_{jk}^+$.

Proof. For a given ω_{jk} we take $\omega \in (\omega_{jk}, \omega_{jk}^R)$ and we first prove that $J''_\omega(u_\omega)$ is negative definite on $\mathbf{R}u_\omega \oplus E_{jk} \oplus \mathbf{R}^N$ provided ω is close to ω_{jk}. This amounts to showing that for every $\alpha \in \mathbf{R}$, every $\varphi \in E_{jk}$, and every $c \in \mathbf{R}^N$ (not all zero)

$$J''_\omega(u_\omega) \cdot (\alpha u_\omega + \varphi + c)^2 < 0.$$

We will do this by expanding every term in

$$J''_\omega(u_\omega) \cdot (\alpha u_\omega + \varphi + c)^2 = \alpha^2 J''_\omega(u_\omega) \cdot u_\omega^2 + J''_\omega(u_\omega) \cdot \varphi^2 + J''_\omega(u_\omega) \cdot c^2$$

$$+2\alpha J''_\omega(u_\omega) \cdot (u_\omega, \varphi) + 2\alpha J''_\omega(u_\omega) \cdot (u_\omega, c) + 2J''_\omega(u_\omega) \cdot (\varphi, c) \qquad (15)$$

as $k \to \infty$ and $\omega \to {\omega_{jk}}^+$ in a sense that we now make precise.
In this section we could work with a fixed k and estimate the terms as $\omega \to \omega_{jk}^+$; however in the next section we will need to take a subsequence $k_n \to \infty$ and, for each n, an ω_n such that $k_n(\omega_n - \omega_{jk_n}) \to 0^+$. Therefore we prefer to make explicit the dependence on k by expressing the estimates from the beginning as asymptotic relations as $k(\omega - \omega_{jk}) \to 0^+$. This, even if not needed in the present section, will spare some computations later. Of course $k(\omega - \omega_{jk}) \to 0^+$ is satisfied when k is fixed and $\omega \to \omega_{jk}^+$.
Thus, every term in (15) will be now estimated asymptotically as $k(\omega - \omega_{jk}) \to 0^+$.
We split the computations in a series of lemmas; assumptions (A), and (W) will be taken for granted without repeating it every time.
In order to simplify notation, we set *in the proofs*

$$s = \omega - \omega_{jk},$$

so that the expansion of Γ_{jk} in the L^∞ norm, given by (14), reads

$$u_\omega = (c_j ks)^{1/2} \varphi_{jk} + O(ks) \tag{16}$$

as $ks \to 0^+$.
The first lemma specifies the asymptotic behavior of some quantities that appear repeatedly in the proofs of this and of the next section. This is the main point where the homogeneity properties of W come into play.

Lemma 3.2 *Let u_ω be as in (14), and let $v, w \in H$. Then*

$$W(u_\omega) = \pi^{-2}(c_j k(\omega - \omega_{jk}))^2 \cos^4(kt) W(e_j) + O((k(\omega - \omega_{jk}))^{\frac{5}{2}}), \tag{17}$$

$$W'(u_\omega) \cdot v = \pi^{-\frac{3}{2}}(c_j k(\omega - \omega_{jk}))^{\frac{3}{2}} \cos^3(kt) W'(e_j) \cdot v + O((k(\omega - \omega_{jk}))^2)|v|, \tag{18}$$

$$W''(u_\omega) \cdot (v, w) = O(k(\omega - \omega_{jk}))|v||w|, \tag{19}$$

where the remainders are in the L^∞ norm.

Proof. By homogeneity of W, it is easily seen using (16) and the definition of φ_{jk} that

$$\begin{aligned} W(u_\omega) &= \frac{1}{4!} W''''(0) \cdot u_\omega^4 = \frac{1}{4!} W''''(0) \cdot ((c_j ks)^{1/2} \varphi_{jk} + O(ks))^4 \\ &= \frac{1}{4!}(c_j ks)^2 \frac{1}{\pi^2} \cos^4(kt) W''''(0) \cdot e_j^4 + O((ks)^{5/2}) \\ &= \frac{1}{\pi^2}(c_j ks)^2 \cos^4(kt) W(e_j) + O((ks)^{5/2}), \end{aligned}$$

which is (17). With the very same argument we see that

$$W'(u_\omega)\cdot v = \frac{1}{6}W''''(0)\cdot(u_\omega^3, v) = \frac{1}{6}W''''(0)\cdot((c_jks)^{1/2}\varphi_{jk}+O(ks))^3, v)$$
$$= \frac{1}{\pi^{3/2}}(c_jks)^{3/2}\cos^3(kt)W'(e_j)\cdot v + |v|\,O((ks)^2),$$

which yields (18). The proof of (19) is even simpler, and we omit it. ∎

We group in the following lemma the estimates of the simplest terms in (15).

Lemma 3.3 *As $k(\omega-\omega_{jk})\to 0^+$ we have*

$$J''_\omega(u_\omega)\cdot u_\omega^2 \le -2\mu_j c_j (k(\omega-\omega_{jk}))^2, \tag{20}$$
$$J''_\omega(u_\omega)\cdot c^2 \le -\pi\mu_1|c|^2, \tag{21}$$
$$J''_\omega(u_\omega)\cdot(u_\omega,\varphi) = O((k(\omega-\omega_{jk}))^{3/2})||\varphi||_2, \tag{22}$$
$$J''_\omega(u_\omega)\cdot(u_\omega,c) = O((k(\omega-\omega_{jk}))^2)|c|, \tag{23}$$
$$J''_\omega(u_\omega)\cdot(\varphi,c) = O(k(\omega-\omega_{jk}))||\varphi||_2\,|c|. \tag{24}$$

Proof. We start by estimating

$$J''(u_\omega)\cdot u_\omega^2 = \int_0^{2\pi} L_\omega u_\omega\cdot u_\omega\,dt - \int_0^{2\pi} W''(u_\omega)\cdot u_\omega^2\,dt.$$

Since u_ω is a solution, and by homogeneity, $L_\omega u_\omega\cdot u_\omega = W'(u_\omega)\cdot u_\omega = 4W(u_\omega)$; still by homogeneity, $W''(u_\omega)\cdot u_\omega^2 = 12W(u_\omega)$. Therefore, by (17) we obtain

$$J''_\omega(u_\omega)\cdot u_\omega^2 = -8\int_0^{2\pi} W(u_\omega)\,dt = -8\left((c_jks)^2\frac{1}{\pi^2}\frac{3\pi}{4}W(e_j)\right) + O((ks)^{5/2})$$
$$= -4\mu_j c_j (ks)^2 + O((ks)^{5/2}),$$

where we have also used the definition of c_j. This (20) for ks small enough.
Passing to (21), with the elementary properties of A and (19) with $v = w = c$, we immediately see that

$$J''_\omega(u_\omega)\cdot c^2 = -2\pi Ac\cdot c - \int_0^{2\pi} W''(u_\omega)\cdot c^2\,dt \le -2\pi\mu_1|c|^2 + |c|^2 O(ks),$$

and we obtain the statement for ks small enough.
We now turn to (22) and (23). Let v be any element of H; since $L_\omega u_\omega = W'(u_\omega)$ and $W''(u_\omega)\cdot(u_\omega,v) = 3W'(u_\omega)\cdot v$, we see that

$$J''_\omega(u_\omega)\cdot(u_\omega,v) = \int_0^{2\pi} W'(u_\omega)\cdot v\,dt - \int_0^{2\pi} W''(u_\omega)\cdot(u_\omega,v)\,dt = -2\int_0^{2\pi} W'(u_\omega)\cdot v\,dt.$$

Applying (18) with $v = \varphi \in E_{jk}$ yields

$$\int_0^{2\pi} W'(u_\omega) \cdot \varphi \, dt = O((ks)^{3/2}) ||\varphi||_2,$$

which proves (22). If $v = c \in \mathbf{R}^N$, then $W'(e_j) \cdot c$ is a constant, so that the vanishing of the integral of $\cos^3(kt)$ cancels the first term of (18), giving

$$\int_0^{2\pi} W'(u_\omega) \cdot c \, dt = O((ks)^2)|c|$$

which proves also (23).
Finally, for (24) we notice that $L_\omega \varphi \in E_{jk}$, namely it is a linear combination of cosines; therefore

$$\int_0^{2\pi} L_\omega \varphi \cdot c \, dt = 0,$$

so that

$$J''_\omega(u_\omega) \cdot (\varphi, c) = \int_0^{2\pi} L_\omega \varphi \cdot c \, dt - \int_0^{2\pi} W''(u_\omega) \cdot (\varphi, c) \, dt = -\int_0^{2\pi} W''(u_\omega) \cdot (\varphi, c) \, dt.$$

Applying (19) with $v = \varphi$ and $w = c$, this last term is $O(ks)||\varphi||_2 \, |c|$, and the proof is complete. ■

We conclude the first set of estimates with the analysis of $J''_\omega(u_\omega) \cdot \varphi^2$.
The behavior of this term, which will play a central role in our argument, depends on the properties of the function defined by

$$\delta(\omega) = \min\{h^2(\omega_{ih}^2 - \omega^2) \mid \omega_{ih} > \omega \}, \qquad 0 < \omega < \mu_N. \tag{25}$$

This function will be estimated from below in the next section. For now we list its simplest properties.

Lemma 3.4 *The function δ is strictly positive in $(0, \mu_N)$; moreover*

$$\lim_{\omega \to \omega_{jk}^+} \delta(\omega) = \delta(\omega_{jk}) \quad \textit{and} \quad \lim_{\omega \to 0^+} \delta(\omega) = 0.$$

Proof. Positivity is obvious. To check the first limit, notice that

$$h^2(\omega_{ih}^2 - \omega_{jk}^2) > h^2(\omega_{ih}^2 - \omega^2) = h^2(\omega_{ih}^2 - \omega_{jk}^2) + O(\omega - \omega_{jk}).$$

Since the equality $\{\omega_{ih} \mid \omega_{ih} > \omega\} = \{\omega_{ih} \mid \omega_{ih} > \omega_{jk}\}$ holds for $\omega \to \omega_{jk}^+$, minimization on this set yields

$$\delta(\omega_{jk}) \geq \delta(\omega) \geq \delta(\omega_{jk}) + O(\omega - \omega_{jk}),$$

which establishes the first limit.
For any given ω, let now $m \in \mathbf{N}$ be such that $\mu_1/(m+1) \leq \omega < \mu_1/m$. Then

$$\delta(\omega) \leq m^2(\omega_{1m}^2 - \omega^2) = m^2(\omega_{1m} - \omega)(\omega_{1m} + \omega) \leq \frac{2\mu_1^2}{m+1},$$

and of course $m \to \infty$ if $\omega \to 0$. ■

We can now conclude the estimates.

Lemma 3.5 *As* $k(\omega - \omega_{jk}) \to 0^+$,

$$J''_\omega(u_\omega) \cdot \varphi^2 \leq -\delta(\omega)||\varphi||_2^2 + O(k(\omega - \omega_{jk}))||\varphi||_2^2.$$

Proof. By definition of L_ω and φ_{ih} we have

$$L_\omega \varphi_{ih} = L_{\omega_{ih}} \varphi_{ih} - (\omega^2 - \omega_{ih}^2)\ddot{\varphi}_{ih} = h^2(\omega^2 - \omega_{ih}^2)\varphi_{ih},$$

since $\varphi_{ih} \in \text{Ker } L_{\omega_{ih}}$.
Therefore, if we write as above $\varphi = \sum_{i,h} a_{ih}\varphi_{ih}$, with the sum extended to all (i,h) such that $\omega_{ih} > \omega_{jk}$, we see that

$$\int_0^{2\pi} L_\omega \varphi \cdot \varphi \, dt = \sum_{i,h} h^2(\omega^2 - \omega_{ih}^2) a_{ih}^2 \leq -\delta(\omega) \sum_{i,h} a_{ih}^2 = -\delta(\omega)||\varphi||_2^2.$$

Then

$$J''_\omega(u_\omega) \cdot \varphi^2 = \int_0^{2\pi} L_\omega \varphi \cdot \varphi \, dt - \int_0^{2\pi} W''(u_\omega) \cdot \varphi^2 \, dt \leq -\delta(\omega)||\varphi||_2^2 + O(ks)||\varphi||_2^2,$$

since the second integral can be treated as we did to analogous terms in the previous lemmas. ∎

Conclusion of the proof of Theorem 3.1. We are now ready to replace all the terms in (15) with their expansions obtained in the preceding lemmas. We find, as $ks \to 0^+$,

$$\begin{aligned} J''_\omega(u_\omega) \cdot (\alpha u_\omega + \varphi + c)^2 &\leq -2\mu_j c_j (ks)^2 \alpha^2 - (\delta(\omega) + O(ks))||\varphi||_2^2 \\ &-\pi\mu_1 |c|^2 + O((ks)^{3/2})|\alpha|||\varphi||_2 + O((ks)^2)|\alpha||c| + O(ks)||\varphi||_2|c| \end{aligned} \tag{26}$$

We think of the right–hand–side as a bilinear form in $(|\alpha|, ||\varphi||_2, |c|)$ represented by the matrix

$$M_3 = \begin{pmatrix} -2\mu_j c_j (ks)^2 & O((ks)^{3/2}) & O((ks)^2) \\ O((ks)^{3/2}) & -\delta(\omega) + O(ks) & O(ks) \\ O((ks)^2) & O(ks) & -\pi\mu_1 \end{pmatrix}.$$

In order to show that M_3 is negative definite we check that $(-1)^\ell \det M_\ell > 0$ for $\ell = 1, 2, 3$, where the M_ℓ's are the principal square submatrices. Now $-\det M_1 = 2\mu_j c_j (ks)^2 > 0$, while

$$\begin{aligned} \det M_2 &= 2\mu_j c_j (ks)^2 [\delta(\omega) + O(ks)] \\ -\det M_3 &= 2\pi\mu_1\mu_j c_j (ks)^2 [\delta(\omega) + O(ks)] \end{aligned}$$

Since for each fixed $k \in \mathbf{N}^+$, when $s \to 0^+$, namely when $\omega \to \omega_{jk}^+$, we have $\delta(\omega) \to \delta(\omega_{jk}) > 0$ by Lemma 3.4 and, of course, $ks \to 0$; therefore $\det M_2$ and $-\det M_3$ are both positive as $\omega \to \omega_{jk}$.
We have thus proved that for every $j = 1, \ldots, N$ and every $k \in \mathbf{N}^+$, the quadratic form $J''_\omega(u_\omega)$ is negative definite on $\mathbf{R}u_\omega \oplus E_{jk} \oplus \mathbf{R}^N$ provided ω is close enough to ω_{jk} (from the right); in other words, for all such ω's,

$$m(u_\omega) \geq \nu_{jk} + N + 1. \tag{27}$$

The proof will be complete when we establish the reversed inequality.
To this aim it suffices to find a subspace of H of codimension $\nu_{jk} + N + 1$ on which $J''_\omega(u_\omega)$ is positive definite. We claim that one such subspace is

$$F_{jk} = \text{ cl span } \{\varphi_{ih} \mid \omega_{ih} < \omega_{jk}\}$$

(closure taken in H). Notice that since $H = F_{jk} \oplus \mathbf{R}\varphi_{jk} \oplus E_{jk} \oplus \mathbf{R}^N$ and $\dim(\mathbf{R}\varphi_{jk} \oplus E_{jk} \oplus \mathbf{R}^N) = \nu_{jk} + N + 1$, the space F_{jk} has the right codimension. We just have to show that $J''_\omega(u_\omega)$ is positive definite on F_{jk}.
Now if $\varphi \in F_{jk}$, then $\varphi = \sum_{i,h} a_{ih}\varphi_{ih}$, where the series is extended to all i, h such that $\omega_{ih} < \omega_{jk}$. Then with the same computations as in Lemma 3.5,

$$\int_0^{2\pi} L_\omega\varphi \cdot \varphi\, dt = \sum_{i,h} h^2(\omega^2 - \omega_{ih}^2)a_{ih}^2 \geq \rho(\omega) \sum_{i,h} a_{ih}^2 = \rho(\omega)||\varphi||_2^2$$

where $\rho(\omega) = \inf\{h^2(\omega^2 - \omega_{ih}^2) \mid \omega_{ih} < \omega_{jk}\}$. Notice that for every $\omega \geq \omega_{jk}$ we have $\rho(\omega) \geq \omega_{jk}^2 - (\omega_{jk}^L)^2$, a number that does not depend on ω. Moreover, with by now standard computations,

$$\int_0^{2\pi} W''(u_\omega) \cdot \varphi^2\, dt = O(ks)||\varphi||_2^2.$$

Therefore

$$J''_\omega(u_\omega) \cdot \varphi^2 \geq ((\omega_{jk}^2 - (\omega_{jk}^L)^2) + O(ks))||\varphi||_2^2;$$

by standard arguments, this is enough to prove that $J''_\omega(u_\omega)$ is positive definite on F_{jk} also for the H topology, for ω close to ω_{jk}. ∎

4 Asymptotic resonance and Morse index jumps

The analysis of the Morse index was carried out in the previous section on a single primary branch, independently of its position among the other branches. From now on we consider *pairs* of primary branches Γ_{ih} and Γ_{jk}, originating at close characteristic values

$$\omega_{ih} < \omega_{jk},$$

with the aim of describing their mutual interaction. More precisely, in this section we will study under which circumstances the presence of Γ_{ih} affects the Morse index along Γ_{jk}; the analysis of the symmetric case will be dealt with in the next section.
The main result will show that for ω "far enough" from ω_{jk} the function φ_{ih} shifts from the positive to the negative eigenspace of $J''_\omega(u_\omega)$. Since this fact cannot happen for linear problems ($W \equiv 0$), it must be an effect of the nonlinearity that ignites the change of the index. In our case it takes the very simple and computable form

$$6\mu_i^2 W(e_j) - \mu_j^2 W''(e_j) \cdot e_i^2 < 0,$$

to which we refer as a *nonlinear coupling* of the frequency μ_i with μ_j. We will study this inequality in detail for the Fermi–Pasta–Ulam model in Section 9.
The key point in this section is the choice of the right ω to compute the index of $J''_\omega(u_\omega)$; notice for example that in order to use the expansion (14) we must work with ω very close to ω_{jk}, while the property we need can only hold for ω far enough from ω_{jk}. The fulfillment of these competing requirements forces us to consider, instead of a fixed pair of characteristic frequencies, *sequences* of pairs

$$\omega_{ih_n} < \omega_{jk_n} \qquad \text{with} \qquad h_n, k_n \to \infty.$$

We now make a preliminary analysis of these sequences, in order to establish the right tuning for $\omega = \omega_n$. Recall that ω^R denotes the smallest element of Ω at the right of ω.
First of all notice that if k_n is a diverging sequence of integers, then

$$\omega^R_{jk_n} - \omega_{jk_n} \leq \frac{\mu_j}{k_n - 1} - \frac{\mu_j}{k_n} = O(\frac{1}{k_n^2}). \tag{28}$$

This means not only that $\omega^R_{jk_n}$ lies in the domain of Γ_{jk_n}, (see Definition 2.2), but also guarantees the stronger property

$$k_n(\omega^R_{jk_n} - \omega_{jk_n}) \to 0^+. \tag{29}$$

In other words, the expansion

$$u_{\omega_n} = (c_j k_n(\omega_n - \omega_{jk_n}))^{1/2}\varphi_{jk_n} + O(k_n(\omega_n - \omega_{jk_n})) \tag{30}$$

makes sense at every ω_n satisfying $\omega_{jk_n} < \omega_n < \omega^R_{jk_n}$.
We next introduce a key concept in our work.

Definition 4.1 *We say that μ_i is* right asymptotically resonant *with μ_j if there exist diverging sequences $h_n, k_n \in \mathbf{N}$ such that*

$$\omega_{ih_n} < \omega_{jk_n} \quad \forall n \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{\omega_{jk_n} - \omega_{ih_n}}{\omega^R_{jk_n} - \omega_{jk_n}} = 0. \tag{31}$$

Notice that the conditions in (31) are not symmetric with respect to μ_i and μ_j. The existence of frequencies that satisfy (31) is determined only by the arithmetical properties of the set of the eigenvalues of the matrix A, and is therefore a property of the *linear* part of the problem. In Section 7 we characterize the asymptotic resonance for a general set of frequencies and in Section 8 we particularize the result to the Fermi–Pasta–Ulam problem. For the moment we simply assume that a frequency μ_i is right asymptotically resonant with μ_j, and we deduce some consequences.
The first one concerns the relations between h_n and k_n hidden in (31).

Lemma 4.2 *Assume condition (31) holds. Then $i \neq j$; moreover for all n large enough, ω_{jk_n} is the first element of Ω_j at the right of ω_{ih_n}; symmetrically, ω_{ih_n} is the first element of Ω_i at the left of ω_{jk_n} and finally*

$$\frac{k_n}{h_n} \to \frac{\mu_j}{\mu_i}. \tag{32}$$

Proof. As already computed, the denominator in (31) is estimated from above by the quantity $\mu_j/k_n(k_n-1)$. On the other hand, if $i = j$, or (along a subsequence) $i \neq j$ but ω_{jk_n} is not the first element of Ω_j at the right of ω_{ih_n}, then

$$\omega_{jk_n} - \omega_{ih_n} \geq \frac{\mu_j}{k_n} - \frac{\mu_j}{k_n+1} = \frac{\mu_j}{k_n(k_n+1)}.$$

In both cases

$$\frac{\omega_{jk_n} - \omega_{ih_n}}{\omega^R_{jk_n} - \omega_{jk_n}} \geq \frac{k_n(k_n-1)}{k_n(k_n+1)} \to 1,$$

violating (31). Therefore $i \neq j$ and

$$\frac{\mu_j}{k_n+1} \leq \frac{\mu_i}{h_n} < \frac{\mu_j}{k_n},$$

from which the limit (32) follows. It remains to prove that ω_{ih_n} is the first element of Ω_i at the left of ω_{jk_n}. Observe indeed that, if (along a subsequence) this is false, then

$$\omega_{jk_n} - \omega_{ih_n} \geq \frac{\mu_i}{h_n-1} - \frac{\mu_i}{h_n} = \frac{\mu_i}{h_n(h_n-1)}.$$

Since we already know that h_n has the same order as k_n, this leads to the same contradiction as before. ■

Remark 4.3 A further consequence of Definition 4.1, via (28), is the fact that

$$\omega_{jk_n} - \omega_{ih_n} = o(\omega^R_{jk_n} - \omega_{jk_n}) = o(\frac{1}{k_n^2}).$$

Thus, although we cannot have an exact resonance involving μ_i and μ_j because of (Q), the preceding relation shows that along the integer sequences k_n and h_n we have

$$h_n\,\mu_j - k_n\,\mu_i \;=\; o(1),$$

namely an "asymptotic resonance". This justifies the term adopted in the above definition.

The introduction of the notion of right asymptotic resonance allows us to complete the required balance between "how close" and "how far" ω_n should be from ω_{jk_n}. Indeed we choose a frequency ω_n such that

$$\omega_{jk_n} < \omega_n < \omega^R_{jk_n}, \tag{33}$$

$$\omega_{jk_n} - \omega_{ih_n} = o(\omega_n - \omega_{jk_n}), \tag{34}$$

$$\omega_n - \omega_{jk_n} \quad = o(\omega^R_{jk_n} - \omega_{jk_n}), \tag{35}$$

simultaneously hold as $n \to \infty$.
Asymptotic resonance guarantees that such choice for ω_n is possible, as for instance the position

$$\omega_n = \omega_{jk_n} + (\omega_{jk_n} - \omega_{ih_n}) \log \frac{\omega^R_{jk_n} - \omega_{jk_n}}{\omega_{jk_n} - \omega_{ih_n}}$$

makes evident.
The sense of the requirements (34) and (35) is most easily visualized if one thinks of the three adjacent intervals determined by

$$\omega_{ih_n} < \omega_{jk_n} < \omega_n < \omega^R_{jk_n}.$$

The choice of ω_n makes the central interval (asymptotically) much larger than the left one and much smaller than the right one.
We will evaluate the Morse index precisely at the points ω_n; in view of future computations, we establish the following results.

Lemma 4.4 *Assume that (31) holds and choose ω_n according to (33) and (34). Then, as $n \to \infty$,*

$$\frac{k_n(\omega_n - \omega_{jk_n})}{h_n(\omega_n - \omega_{ih_n})} \to \frac{\mu_j}{\mu_i}\;.$$

Proof. Using Lemma 4.2 compute

$$\frac{k_n(\omega_n - \omega_{jk_n})}{h_n(\omega_n - \omega_{ih_n})} = \frac{k_n}{h_n}\,\frac{\omega_n - \omega_{jk_n}}{(\omega_n - \omega_{jk_n}) + (\omega_{jk_n} - \omega_{ih_n})} = \left(\frac{\mu_j}{\mu_i} + o(1)\right)\,\frac{1}{1+o(1)}.$$

■

The second property provides an estimate from below of the rate of vanishing of the function δ, defined by (25), when computed at ω_n.

Lemma 4.5 *If ω_n is chosen according to (33) and (35) then, as $n \to \infty$,*

$$\frac{\delta(\omega_n)}{k_n(\omega_n - \omega_{jk_n})} \to +\infty.$$

Proof. For every $\alpha = 1, \ldots, N$, denote by $\mu_\alpha/p_{\alpha n}$ the first element of Ω_α at the right of ω_{jk_n}; by (33) it also the first element of Ω_α at the right of ω_n. It is easy to see that $p_{\alpha n} \asymp k_n$ as $n \to \infty$.
Notice that for fixed α and n, the function $p \mapsto \mu_\alpha^2 - p^2\omega_n^2$ is decreasing. Therefore

$$\delta(\omega_n) = \min\{\mu_\alpha^2 - p_{\alpha n}^2\omega_n^2 \mid \alpha = 1, \ldots, N\}.$$

Now for every $\alpha = 1, \ldots, N$,

$$\begin{aligned} \mu_\alpha^2 - p_{\alpha n}^2\omega_n^2 &= p_{\alpha n}^2(\mu_\alpha/p_{\alpha n} - \omega_n)(\mu_\alpha/p_{\alpha n} + \omega_n) \\ &\geq 2\omega_{jk_n}p_{\alpha n}^2(\omega_{jk_n}^R - \omega_n) \asymp k_n(\omega_{jk_n}^R - \omega_n). \end{aligned}$$

Dividing by $k_n(\omega_n - \omega_{jk_n})$ and using (35) we obtain the required estimate. ∎

We are now ready to state the main result of this section. In its formulation notice the two assumptions involving the linear and nonlinear phenomena discussed earlier.

Theorem 4.6 *Assume that (A) and (W) hold, and suppose moreover that*

1) μ_i *is right asymptotically resonant with* μ_j*,*

2) $6\mu_i^2 W(e_j) - \mu_j^2 W''(e_j) \cdot e_i^2 < 0\,.$

Then, for every choice of h_n and k_n according to (31), and every choice of ω_n according to (33), (34) and (35), we have

$$m(u_{\omega_n}) \geq \nu_{jk_n} + N + 2$$

for every n large enough.

Proof. We follow closely the proof of Theorem 3.1, in the sense that with a series of estimates we will show that, for n large, the quadratic form $J''_{\omega_n}(u_{\omega_n})$ is negative definite on the subspace of H given by $\mathbf{R}u_{\omega_n} \oplus \mathbf{R}\varphi_{ih_n} \oplus E_{jk_n} \oplus \mathbf{R}^N$. This is easily seen to have dimension $\nu_{jk_n} + N + 2$.
The relation to check is thus

$$J''_{\omega_n}(u_{\omega_n}) \cdot (\alpha u_{\omega_n} + \beta\varphi_{ih_n} + \varphi + c)^2 < 0$$

for n large, and for every $\alpha, \beta \in \mathbf{R}$, every $\varphi \in E_{jk_n}$, and every $c \in \mathbf{R}^N$ (not all zero).
Writing

$$\begin{aligned} J''_{\omega_n}(u_{\omega_n})\cdot&(\alpha u_{\omega_n} + \beta\varphi_{ih_n} + \varphi + c)^2 \\ &= J''_{\omega_n}(u_{\omega_n}) \cdot (\alpha u_{\omega_n} + \varphi + c)^2 + \beta^2 J''_{\omega_n}(u_{\omega_n}) \cdot \varphi^2_{ih_n} \\ &+2\beta J''_{\omega_n}(u_{\omega_n}) \cdot (\varphi_{ih_n}, \varphi) + 2\beta J''_{\omega_n}(u_{\omega_n}) \cdot (\varphi_{ih_n}, c) \\ &+2\alpha\beta J''_{\omega_n}(u_{\omega_n}) \cdot (u_{\omega_n}, \varphi_{ih_n}), \end{aligned} \tag{36}$$

we see that the first term in the right–hand–side has already been estimated in (26), so that we can concentrate on the remaining four terms.
As in the proof of Theorem 3.1 we divide the computations in a series of lemmas, where the conditions (A) and (W) are taken for granted, without repeating it every time.
In the proofs of these lemmas, as we did in the previous section to simplify notation, we will set

$$\varepsilon_n = \omega_{jk_n} - \omega_{ih_n}, \qquad \Delta_n = \omega^R_{jk_n} - \omega_{jk_n}, \qquad s_n = \omega_n - \omega_{jk_n}, \qquad \sigma_n = \omega_n - \omega_{ih_n}.$$

With these conventions, right asymptotic resonance takes the form $\varepsilon_n = o(\Delta_n)$, while the requirements on ω_n are expressed by

$$0 < s_n < \Delta_n, \qquad \varepsilon_n = o(s_n), \qquad s_n = o(\Delta_n)\ .$$

Moreover, the expansions of Γ_{jk_n} reads

$$u_\omega = (c_j k_n s_n)^{1/2}\varphi_{jk_n} + O(k_n s_n) \qquad \text{as} \qquad k_n s_n \to 0^+,$$

and, due to Lemma 4.4,

$$\frac{k_n s_n}{h_n \sigma_n} \ \to\ \frac{\mu_j}{\mu_i}\ .$$

This last relation allows us to substitute $h_n\sigma_n$ (to any power) with $O(k_n s_n)$ (to the same power) in the computations below.
Finally we notice that

$$L_{\omega_n}\varphi_{ih_n} = h_n^2(\omega_n^2 - \omega_{ih_n}^2)\,\varphi_{ih_n} = h_n\sigma_n(2\mu_i + h_n\sigma_n)\,\varphi_{ih_n}, \tag{37}$$

which will be used repeatedly.
We begin with the estimate of the mixed terms, which are easier.

Lemma 4.7 *As $k(\omega - \omega_{jk}) \to 0^+$ we have*

$$J''_{\omega_n}(u_{\omega_n}) \cdot (\varphi_{ih_n}, \varphi) = O(k_n(\omega_n - \omega_{jk_n}))||\varphi||_2, \tag{38}$$

$$J''_{\omega_n}(u_{\omega_n}) \cdot (\varphi_{ih_n}, c) = O(k_n(\omega_n - \omega_{jk_n}))|c|, \tag{39}$$

$$J''_{\omega_n}(u_{\omega_n}) \cdot (u_{\omega_n}, \varphi_{ih_n}) = O((k_n(\omega_n - \omega_{jk_n}))^2). \tag{40}$$

Proof. Concerning (38) and (39), notice that for every $w \in H$ we have

$$\int_0^{2\pi} L_{\omega_n}\varphi_{ih_n} \cdot w\, dt = h_n\sigma_n(2\mu_i + h_n\sigma_n)\int_0^{2\pi} \varphi_{ih_n} \cdot w\, dt = O(k_n s_n)||w||_2,$$

while using (19) we obtain

$$\int_0^{2\pi} W''(u_{\omega_n}) \cdot (\varphi_{ih_n}, w)\, dt = O(k_n s_n)||w||_2.$$

To prove the last statement, since u_{ω_n} is a solution, and by homogeneity,

$$J''_{\omega_n}(u_{\omega_n}) \cdot (u_{\omega_n}, \varphi_{ih_n}) = -2\int_0^{2\pi} W'(u_{\omega_n}) \cdot \varphi_{ih_n}\, dt.$$

Now, using (18), we obtain

$$\begin{aligned}&\int_0^{2\pi} W'(u_{\omega_n}) \cdot \varphi_{ih_n}\, dt\\ &= \frac{1}{\pi^2}(c_j k_n s_n)^{\frac{3}{2}}(W'(e_j) \cdot e_i)\int_0^{2\pi} \cos^3(k_n t)\cos(h_n t)\, dt + O((k_n s_n)^2)\end{aligned}$$

Since $k_n/h_n \to \mu_j/\mu_i \notin \mathbf{Q}$, we know that for all n large, $h_n \neq k_n$ and $h_n \neq 3k_n$. Therefore

$$\int_0^{2\pi} \cos^3(k_n t)\cos(h_n t)\, dt = \frac{1}{4}\int_0^{2\pi} (\cos(3k_n t) + 3\cos(k_n t))\cos(h_n t)\, dt = 0$$

for all large n, and (40) follows. ∎

We conclude this set of estimates with the analysis of $J''_{\omega_n}(u_{\omega_n}) \cdot \varphi^2_{ih_n}$. As we anticipated, this term plays a central role, since it is where the nonlinear coupling coefficient

$$W_{ij} = 6\mu_i^2 W(e_j) - \mu_j^2 W''(e_j) \cdot e_i^2 \tag{41}$$

comes into play.

Lemma 4.8 *As* $n \to \infty$,

$$J''_{\omega_n}(u_{\omega_n}) \cdot \varphi^2_{ih_n} = \frac{W_{ij}}{3\mu_j W(e_j)}(k_n(\omega_n - \omega_{jk_n})) + o(k_n(\omega_n - \omega_{jk_n})).$$

Proof. By (37) we have

$$\int_0^{2\pi} L_{\omega_n}\varphi_{ih_n} \cdot \varphi_{ih_n}\, dt = h_n\sigma_n(2\mu_i + h_n\sigma_n),$$

while, with the usual arguments,

$$\int_0^{2\pi} W''(u_{\omega_n}) \cdot \varphi_{ih_n}^2 \, dt = \frac{1}{2}\int_0^{2\pi} W''''(0) \cdot (u_{\omega_n}^2, \varphi_{ih_n}^2) \, dt$$
$$= \frac{1}{\pi^2}(c_j k_n s_n) W''(e_j) \cdot e_i^2 \int_0^{2\pi} \cos^2(k_n t) \cos^2(h_n t) \, dt + O((k_n s_n)^{3/2}).$$

Since the integral equals $\pi/2$ unless $h_n = k_n$, which we can exclude for n large by (32) and (Q), we obtain

$$J''_{\omega_n}(u_{\omega_n}) \cdot \varphi_{ih_n}^2 = h_n\sigma_n(2\mu_i + h_n\sigma_n) - \frac{1}{2\pi}(c_j k_n s_n) W''(e_j) \cdot e_i^2 + O((k_n s_n)^{3/2}).$$

Recalling that $h_n\sigma_n = k_n s_n \mu_i/\mu_j + o(1)$ and substituting, we find

$$J''_{\omega_n}(u_{\omega_n}) \cdot \varphi_{ih_n}^2 = \Big(2\frac{\mu_i^2}{\mu_j} - \frac{1}{2\pi} c_j W''(e_j) \cdot e_i^2\Big) k_n s_n + o(k_n s_n)$$
$$= \frac{1}{3\mu_j W(e_j)}\Big(6\mu_i^2 W(e_j) - \mu_j^2 W''(e_j) \cdot e_i^2\Big) k_n s_n + o(k_n s_n),$$

where we have used the definition of c_j for the last equality. This completes the proof and the series of estimates we need. ■

For further use, we notice that since the coefficient of the principal part of the term $J''_{\omega_n}(u_{\omega_n}) \cdot \varphi_{ih_n}^2$ is negative by assumption, for $k_n s_n$ small we can write

$$J''_{\omega_n}(u_{\omega_n}) \cdot \varphi_{ih_n}^2 \leq N_{ij}(k_n s_n),$$

where

$$N_{ij} = \frac{W_{ij}}{6\mu_j W(e_j)} < 0.$$

Conclusion of the proof of Theorem 4.6. We are ready to replace every term in (36) with its asymptotic expansion. For the last four terms we use the preceding lemmas, while for $J''_{\omega_n}(u_{\omega_n}) \cdot (\alpha u_{\omega_n} + \varphi + c)^2$ we use the results of Section 3, given in (26), and particularized here to $k = k_n$, $\omega = \omega_n$, so that also $s = s_n$. We obtain, as $k_n s_n \to 0^+$,

$$J''_{\omega_n}(u_{\omega_n}) \cdot (\alpha u_{\omega_n} + \beta\varphi_{ih_n} + \varphi + c)^2 \leq -2\mu_j c_j (k_n s_n)^2 \alpha^2 - (\delta(\omega) + O(k_n s_n))||\varphi||_2^2$$
$$-\pi\mu_1 |c|^2 + O((k_n s_n)^{3/2})|\alpha|||\varphi||_2 + O((k_n s_n)^2)|\alpha||c| + O(k_n s_n)||\varphi||_2|c|$$
$$+\beta^2 N_{ij}(k_n s_n) + O(k_n s_n)|\beta|||\varphi||_2 + O((k_n s_n)^2)|\alpha||\beta| + O(k_n s_n)|\beta||c|$$

which we consider as a bilinear form in $(|\alpha|, ||\varphi||_2, |c|, |\beta|)$, represented by the matrix

$$M_4 = \begin{pmatrix} -2\mu_j c_j (k_n s_n)^2 & O((k_n s_n)^{3/2}) & O((k_n s_n)^2) & O((k_n s_n)^2) \\ O((k_n s_n)^{3/2}) & -\delta(\omega) + O(k_n s_n) & O(k_n s_n) & O(k_n s_n) \\ O((k_n s_n)^2) & O(k_n s_n) & -\pi\mu_1 & O(k_n s_n) \\ O((k_n s_n)^2) & O(k_n s_n) & O(k_n s_n) & N_{ij}(k_n s_n) \end{pmatrix}.$$

We now show that M_4 is negative definite. Once again we prove that the principal square submatrices M_ℓ satisfy $(-1)^\ell \det M_\ell > 0$, this time for $\ell = 1, \ldots, 4$. We take advantage of the computations carried out in the previous section, which are valid for any j, k and ω.
Obviously $-\det M_1 = 2\mu_j c_j (k_n s_n)^2 > 0$. By the results of the previous section we have

$$\det M_2 = 2\mu_j c_j (k_n s_n)^2 [\delta(\omega_n) + O(k_n s_n)] = 2\mu_j c_j (k_n s_n)^3 [\frac{\delta(\omega_n)}{k_n s_n} + O(1)]$$

$$-\det M_3 = 2\pi\mu_1 \mu_j c_j (k_n s_n)^2 [\delta(\omega_n) + O(k_n s_n)] = 2\pi\mu_1 \mu_j c_j (k_n s_n)^3 [\frac{\delta(\omega_n)}{k_n s_n} + O(1)],$$

which are both positive, for all large n, by Lemma 4.5.
The evaluation of $\det M_4$ is rather boring, though elementary. We expand $\det M_4$ along the last column denoting by $M_{\iota\kappa}$ the matrix obtained from M_4 erasing the ι–th row and the κ–th column. With straightforward computations we obtain

$$\det M_{14} = O(k_n s_n)^{5/2} + O(k_n s_n)^2 (\delta(\omega_n) + O(k_n s_n));$$

since $\delta(\omega_n) \to 0$ as $n \to \infty$, we immediately recognize that

$$\det M_{14} = O(k_n s_n)^2.$$

Next, without even using properties of δ we obtain

$$\det M_{24} = O(k_n s_n)^3$$

and, with the same argument as for M_{14},

$$\det M_{34} = O(k_n s_n)^3.$$

Finally, we can read $\det M_{44}$ from the results of the previous section, to find

$$\det M_{44} = -2\pi\mu_1 \mu_j c_j (k_n s_n)^2 [\delta(\omega_n) + O(k_n s_n)].$$

We obtain therefore the estimate of $\det M_4$

$$\det M_4 = -2\pi\mu_1\mu_j c_j N_{ij}(k_n s_n)^3[\delta(\omega_n) + O(k_n s_n)] + O(k_n s_n)^4$$

Since this can be written

$$\det M_4 = -2\pi\mu_1\mu_j c_j N_{ij}(k_n s_n)^4[\frac{\delta(\omega_n)}{k_n s_n} + O(1)],$$

using Lemma 4.5 and recalling that by assumption $N_{ij} < 0$, we see that $\det M_4$ is positive for all n large enough.
These sign relations prove that the quadratic form $J''_{\omega_n}(u_{\omega_n})$ is negative definite on $\mathbf{R}u_{\omega_n} \oplus \mathbf{R}\varphi_{ih_n} \oplus E_{jk_n} \oplus \mathbf{R}^N$, provided $\omega_n = \omega_{jk_n} + s_n \in (\omega_{jk_n}, \omega^R_{jk_n})$ is chosen according to the procedure described. We have therefore obtained that

$$m(u_{\omega_n}) \geq \nu_{jk_n} + N + 2,$$

and the proof is complete. ∎

5 Bifurcation from the left branch

The aim of this section is to carry out an analysis of the Morse index on the left branches (those departing from ω_{ih}) similar to the one developed in the previous section for the right branches. Here, instead of repeating computations which are analogous to the ones of the previous section, we will just point out the differences, the main one being that we now expect the Morse index to *decrease* along the branches.
¿From now on, the assumptions (A) and (W) are everywhere taken for granted.
Similarly to Definition 4.1 we now have

Definition 5.1 *We say that* μ_i *is* left asymptotically resonant *with* μ_j *if there exist diverging sequences* $h_n, k_n \in \mathbf{N}$ *such that*

$$\omega_{ih_n} < \omega_{jk_n} \quad \forall n \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{\omega_{jk_n} - \omega_{ih_n}}{\omega_{ih_n} - \omega^L_{ih_n}} = 0. \tag{42}$$

It is easy to see that the conclusions of Lemma 4.2 hold again with the very same proof.
Denoting by v_ω the parametrization of Γ_{ih_n} we notice that, for every fixed n, the arguments of Section 3, and in particular Theorem 3.1, apply to show that

$$m(v_\omega) = \nu_{ih_n} + N + 1$$

as $\omega \to \omega^+_{ih_n}$, where we recall that $\nu_{ih_n} = \dim \text{span } \{\varphi_{\beta q} \mid \omega_{\beta q} > \omega_{ih_n}\}$.

Next we show that, provided the correct nonlinear coupling condition is satisfied, condition (42) allows us to choose ω_n such that

$$J''_{\omega_n}(v_{\omega_n}) > 0 \qquad \text{on} \qquad F_{ih_n} \oplus \mathbf{R}\varphi_{jk_n}, \tag{43}$$

where $F_{ih_n} = \text{cl span}\{\varphi_{\beta q} \mid \omega_{\beta q} < \omega_{ih_n}\}$, thereby proving that the Morse index of v_ω decreases to $\nu_{ih_n} + N$ when ω passes from ω_{ih_n} to ω_n. As in the previous section, the right choice of ω_n is obtained by balancing competing requirements, which now lead to the conditions

$$\omega_{ih_n} < \omega_n < \omega_{ih_n} + (\omega_{ih_n} - \omega^L_{ih_n}), \tag{44}$$
$$\omega_{jk_n} - \omega_{ih_n} = o(\omega_n - \omega_{ih_n}), \tag{45}$$
$$\omega_n - \omega_{ih_n} \quad = o(\omega_{ih_n} - \omega^L_{ih_n}). \tag{46}$$

which are satisfied, because of (42), for example by setting

$$\omega_n = \omega_{ih_n} + (\omega_{jk_n} - \omega_{ih_n}) \log \frac{\omega_{ih_n} - \omega^L_{ih_n}}{\omega_{jk_n} - \omega_{ih_n}}.$$

We also notice that (44) implies $h_n(\omega_n - \omega_{ih_n}) \to 0^+$, which guarantees the validity of the expansion

$$v_{\omega_n} = (c_i h_n(\omega_n - \omega_{ih_n}))^{1/2}\varphi_{ih_n} + O(h_n(\omega_n - \omega_{ih_n})).$$

Moreover, in order to prove that $J''_{\omega_n}(v_{\omega_n}) \cdot \varphi_{jk_n} > 0$ we need that $\omega_n > \omega_{jk_n}$. This last inequality follows from (45), which yields

$$\omega_n - \omega_{jk_n} = (\omega_n - \omega_{ih_n}) + (\omega_{ih_n} - \omega_{jk_n}) \sim \omega_n - \omega_{ih_n};$$

therefore $\omega_n - \omega_{jk_n}$ is a positive quantity for n large. Another consequence of the choice of ω_n is that the conclusion of Lemma 4.4 holds again.
The last remark concerns the relative position of ω_n and $\omega^R_{jk_n}$: depending on how close $\omega^R_{jk_n}$ is to ω_{jk_n}, it may possibly be $\omega_n > \omega^R_{jk_n}$. In such case the secondary bifurcation on Γ_{ih_n} may appear after the branch has gone beyond a number of other branches. This is in contrast with the picture for the right branch Γ_{jk_n}, where the secondary bifurcation appears *before* the birth of other branches.

The analogue of the function δ used in the previous section is now given by

$$\rho_n(\omega) = \inf\{p^2(\omega^2 - \omega^2_{\alpha p}) \mid \omega_{\alpha p} < \omega_{ih_n}\}.$$

The key estimate is provided by the following lemma.

Lemma 5.2 *If ω_n is chosen according to (44) and (46) then, as $n \to \infty$,*

$$\frac{\rho_n(\omega_n)}{h_n(\omega_n - \omega_{ih_n})} \to +\infty\ .$$

Proof. For every $\alpha = 1, \ldots, N$, denote by $\mu_\alpha/q_{\alpha n}$ the first element of Ω_α at the left of ω_{ih_n}. It is easy to check that $q_{\alpha n} \asymp h_n$ as $n \to \infty$.
Notice that for fixed α and n, the function $q \mapsto q^2\omega_n^2 - \mu_\alpha^2$ is increasing. Therefore

$$\rho_n(\omega_n) = \min\{q_{\alpha n}^2\, \omega_n^2 - \mu_\alpha^2 \mid \alpha = 1, \ldots, N\}.$$

Now for every $\alpha = 1, \ldots, N$,

$$\begin{aligned} q_{\alpha n}^2\, \omega_n^2 - \mu_\alpha^2 &= q_{\alpha n}^2(\omega_n - \mu_\alpha/q_{\alpha n})(\omega_n + \mu_\alpha/q_{\alpha n}) \\ &> 2\mu_\alpha q_{\alpha n}(\omega_{ih_n} - \omega_{ih_n}^L) \asymp h_n(\omega_{ih_n} - \omega_{ih_n}^L). \end{aligned}$$

Dividing by $h_n(\omega_n - \omega_{ih_n})$ and using (46) we obtain the required estimate. ■

We are finally ready to state the main result of the section.

Theorem 5.3 *Assume that* (A) *and* (W) *hold, and suppose moreover that*

1) μ_i *is left asymptotically resonant with* μ_j*,*

2) $6\mu_j^2 W(e_i) - \mu_i^2 W''(e_i) \cdot e_j^2 > 0\,.$

Then, for every choice of h_n *and* k_n *according to (42), and every choice of* ω_n *according to (44), (45) and (46), we have*

$$m(v_{\omega_n}) \leq \nu_{ih_n} + N$$

for every n *large enough.*

Proof. We only sketch the proof, since it depends on the very same arguments used in the previous section. Denoting by ψ the generic element of F_{ih_n}, one may check that

$$\begin{aligned} &J''_{\omega_n}(v_{\omega_n}) \cdot \psi^2 \geq [\rho_n(\omega_n) + O(h_n(\omega_n - \omega_{ih_n}))]\ ||\psi||_2^2, \\ &J''_{\omega_n}(v_{\omega_n}) \cdot (\psi, \varphi_{ih_n}) = O(h_n(\omega_n - \omega_{ih_n})). \end{aligned}$$

Moreover, using (45), one obtains

$$J''_{\omega_n}(v_{\omega_n}) \cdot \varphi_{jk_n}^2 = \frac{W_{ji}}{3\mu_i W(e_i)}\ (h_n(\omega_n - \omega_{ih_n})) + o(h_n(\omega_n - \omega_{ih_n})),$$

where

$$W_{ji} = 6\mu_j^2 W(e_i) - \mu_i^2 W''(e_i) \cdot e_j^2 > 0.$$

The positivity of $J''_{\omega_n}(v_{\omega_n})$ on $F_{ih_n} \oplus \mathbf{R}\varphi_{jk_n}$ then follows from Lemma 5.2. ■

6 Proof of the main results

In this section we deduce the results stated in the Introduction from Theorems 3.1 and 4.6, already proved. To this aim we will use the structure of the Γ_j's described in Section 2, to which we refer for notation and properties.

Proof of Theorems 1.1 and 1.2. For definiteness, we prove Theorem 1.1; the same argument works for the other result.
To say that the theorem is false means that there exist i, j such that μ_i is right asymptotically resonant with μ_j, but there are no bifurcation points on Γ_{jk_n} for $\omega \in (\omega_{jk_n}, \omega^R_{jk_n})$. Here k_n is the sequence provided by Definition 4.1. A well–known result of Kielhöfer ([10], see also [5]) implies that if at some ω_0 there results

$$\lim_{\omega \to \omega_0^-} m(u_\omega) \neq \lim_{\omega \to \omega_0^+} m(u_\omega),$$

then (ω_0, u_{ω_0}) is either a turning point or a bifurcation point. In our case the curve Γ_{jk_n} is free of turning points in $(\omega_{jk_n}, \omega^R_{jk_n})$ by construction. Moreover since the dependence on ω is analytic, the Morse index along Γ_{jk_n} is locally constant where it is defined. For all ω close to ω_{jk_n} we have $m(u_\omega) = \nu_{jk_n} + N + 1$, by Theorem 3.1; however, by Theorem 4.6 we know that at some $\omega_n \in (\omega_{jk_n}, \omega^R_{jk_n})$ there results $m(u_\omega) \geq \nu_{jk_n} + N + 2$. Therefore there must be a point in $(\omega_{jk_n}, \omega_n)$ where the left and right limits of the $m(u_\omega)$ are different. This is therefore a bifurcation point, contradicting the assumption. ∎

We now turn to the proof of the Birkhoff–Lewis type result, namely Corollary 1.4. This is where the subharmonic nature of the secondary bifurcations becomes manifest. In what follows we denote by $T(f)$ the minimal period of a continuous nonconstant periodic function f.

Proof of Corollary 1.4. We carry out the proof for right asymptotic resonance only. The notation and the properties listed in Section 2 will be used repeatedly, and we set

$$\Lambda_{jk} = \{\pm u_\omega \mid u_\omega \in \Gamma_{jk}\}.$$

Let then i, j be such that μ_i is right asymptotically resonant with μ_j, and let k_n be the corresponding divergent sequence as in Definition 4.1. By Theorem 1.1 we know that for every n (up to subsequences) there is a bifurcation point $(\omega_n, u_n) \in \Gamma_{jk_n}$, with ω_n satisfying

$$k_n(\omega_n - \omega_{jk_n}) \to 0. \tag{47}$$

This means that for every n we can choose

$$(\theta_n, v_n) \in B_{r/k_n}(\omega_{jk_n}) \times B_{k_n R}(0)$$

such that
- $(\theta_n, \pm v_n) \notin \Lambda_{jk_n}$;
- v_n is a solution of problem (P_{θ_n});
- $||v_n - u_n|| + k_n|\theta_n - \omega_n| \to 0$.
Moreover, since u_n has minimal period $2\pi/k_n$ by construction (it lies on Γ_{jk_n}), it is easy to see that (θ_n, v_n) can be taken so close to (ω_{jk_n}, u_n) that

$$T(v_n) = p_n T(u_n) = p_n \frac{2\pi}{k_n} \tag{48}$$

for some $p_n \in \mathbf{N}^+$. Notice that since v_n is 2π–periodic, p_n must divide k_n.
We prove that the sequence x_n of solutions of (1) defined by $x_n(t) = v_n(\theta_n t)$ satisfies the conclusions of the corollary.
First of all we notice that

$$||x_n||_\infty = ||v_n||_\infty \leq ||u_n||_\infty + ||v_n - u_n||_\infty \to 0$$

via the usual expansion (30), since $k_n(\omega_n - \omega_{jk_n}) \to 0$; therefore $x_n \to 0$ in $\mathcal{C}^2$.
We now show that $T(x_n) \to \infty$. If this is false, then up to subsequences

$$T(x_n) = \frac{1}{\theta_n} T(v_n) = \frac{2\pi p_n}{\theta_n k_n}$$

is bounded. By (47) we see that $k_n \omega_n \to \mu_j$, so that by our choice of θ_n we deduce that $k_n \theta_n \to \mu_j$; hence p_n must be bounded. Still passing to subsequences, if necessary, we can assume that $p_n = p \in \mathbf{N}^+$ for all n.
Consider then the functions

$$w_n(t) = v_n(\frac{p}{k_n} t).$$

These are 2π–periodic functions by (48) and are solutions to problems $(P_{\theta_n k_n/p})$, with $\frac{\theta_n k_n}{p} \to \frac{\mu_j}{p}$. Plainly, $||w_n|| \to 0$. Thus we see that

$$(\frac{\theta_n k_n}{p}, w_n) \to (\frac{\mu_j}{p}, 0) \quad \text{in} \quad \mathbf{R} \times H.$$

By the uniqueness properties in Proposition 2.1, it must be

$$(\frac{\theta_n k_n}{p}, w_n) \in \Lambda_{jp}$$

for every n large enough. This means, scaling (recall that p divides k_n), that $(\theta_n, v_n) \in \Lambda_{jk_n}$, which contradicts the initial choice. Therefore, $p_n \to \infty$, and so does $T(x_n)$. ■

7 The notion of asymptotic resonance

This section is devoted to the study of the abstract notions of right and left asymptotic resonance, with the aim of deriving necessary and sufficient conditions for their validity. These conditions will be obtained with the aid of some number theoretical arguments. In the next section we will use the conditions to determine completely the set of asymptotically resonant frequencies in the Fermi–Pasta–Ulam problem.
For the time being we consider a set of $N \geq 2$ positive frequencies

$$\mu_1 < \ldots < \mu_N$$

under the only assumption (Q), namely that they are pairwise independent over the rationals.
We consider also a stronger notion, that combines the ones introduced in Definition 4.1 and 5.1. In its statement, $a \wedge b = \min(a, b)$.

Definition 7.1 *We say that μ_i is* asymptotically resonant *with μ_j if there exist diverging sequences $h_n, k_n \in \mathbf{N}$ such that*

$$\omega_{ih_n} < \omega_{jk_n} \quad \forall n \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{\omega_{jk_n} - \omega_{ih_n}}{(\omega^R_{jk_n} - \omega_{jk_n}) \wedge (\omega_{ih_n} - \omega^L_{ih_n})} = 0. \qquad (49)$$

Roughly speaking, the definition asserts that the characteristic values in the interval $[\omega_{ih_n}, \omega_{jk_n}]$ are (asymptotically) isolated from the remaining part of Ω; this is the case in most of the situations we will describe later on.
The first lemma rewrites the various notions of asymptotic resonance in a more suitable form for the computations.

Lemma 7.2 *Let $N \geq 3$ and assume that (Q) holds. Let two indexes $i \neq j$ be given. Then μ_i is right asymptotically resonant with μ_j if and only if there exists a sequence of positive integers h_n such that for all $\alpha \neq i, j$,*

$$\lim_{n\to\infty} \frac{\left\{ h_n \dfrac{\mu_j}{\mu_i} \right\}}{\left\{ k_n \dfrac{\mu_\alpha}{\mu_j} \right\}} = 0, \qquad (50)$$

where

$$k_n = \left\lfloor h_n \frac{\mu_j}{\mu_i} \right\rfloor. \qquad (51)$$

Left asymptotic resonance is equivalent to require the existence of a sequence h_n such that, for every $\alpha \neq i, j$,

$$\lim_{n\to\infty} \frac{\left\{h_n \frac{\mu_j}{\mu_i}\right\}}{1 - \left\{h_n \frac{\mu_\alpha}{\mu_i}\right\}} = 0, \tag{52}$$

while for asymptotic resonance the two above limit conditions must hold with the same sequence h_n.

Notice that the validity of (50) or (52) yields in particular

$$\lim_{n\to\infty} \left\{h_n \frac{\mu_j}{\mu_i}\right\} = 0, \tag{53}$$

which, contrary to (50) and (52), makes sense also for $N = 2$. It will be clear from the proof that when $N = 2$ the latter condition is equivalent to the fact that μ_i is asymptotically resonant with μ_j, in any of the given definitions. Since μ_j/μ_i is irrational, this condition can always be satisfied along suitable sequences of integers. Summing up, when $N = 2$ and the two frequencies have irrational ratio, then each is asymptotically resonant with the other.

Proof. We give the details only for right asymptotic resonance, the approach being the same in other cases.
For a given μ_i/h, let μ_j/k_h be the first element of Ω_j at its right; because of (Q), it must be

$$k_h = \left\lfloor h\, \frac{\mu_j}{\mu_i} \right\rfloor .$$

Likewise, defining

$$p_{\alpha h} = \begin{cases} \left\lfloor k_h \frac{\mu_\alpha}{\mu_j} \right\rfloor & \text{if } \alpha \neq j, \\ \\ k_h - 1 & \text{if } \alpha = j, \end{cases}$$

we see that $\mu_\alpha/p_{\alpha h}$ is the smallest element of Ω_α at the right of μ_j/k_h, so that

$$\left(\frac{\mu_j}{k_h}\right)^R = \min_{\alpha=1,\dots,N} \frac{\mu_\alpha}{p_{\alpha h}}.$$

If we now introduce the distances

$$\varepsilon_h = \frac{\mu_j}{k_h} - \frac{\mu_i}{h} \qquad \text{and} \qquad \Delta_{\alpha h} = \frac{\mu_\alpha}{p_{\alpha h}} - \frac{\mu_j}{k_h},$$

then Lemma 4.2 says that μ_i is asymptotically resonant with μ_j exactly when

$$\frac{\varepsilon_h}{\Delta_{\alpha h}} \to 0 \qquad \forall \alpha = 1, \ldots, N \tag{54}$$

along some suitable diverging sequence $h = h_n$. This suggests that the proof of the statement depends on the asymptotic behavior of ε_h and $\Delta_{\alpha h}$ as $h \to +\infty$, which we now determine. For writing convenience we set

$$\bar{\mu}_\alpha = \mu_\alpha/\mu_i \qquad \text{and} \qquad \tilde{\mu}_\alpha = \mu_\alpha/\mu_j.$$

First notice that since $k_h \sim h\bar{\mu}_j$, we have

$$\varepsilon_h = \frac{\mu_i}{h\, k_h} (h\bar{\mu}_j - k_h) \sim \frac{\mu_i^2}{\mu_j} \frac{1}{h^2} \{h\bar{\mu}_j\} \qquad \text{and} \qquad \Delta_{jh} = \frac{\mu_j}{k_h\,(k_h - 1)} \sim \frac{\mu_i^2}{\mu_j} \frac{1}{h^2},$$

which yield

$$\frac{\varepsilon_h}{\Delta_{jh}} \sim \{h\bar{\mu}_j\}\,. \tag{55}$$

By (54) this shows that (53) holds.
When $\alpha \neq j$, similar arguments assure that $p_{\alpha h} \sim k_h\tilde{\mu}_\alpha \sim h\bar{\mu}_\alpha$, so that

$$\Delta_{\alpha h} = \frac{\mu_j}{k_h\, p_{\alpha h}} (k_h\, \tilde{\mu}_\alpha - p_{\alpha h}) \sim \frac{\mu_i^2}{\mu_\alpha} \frac{1}{h^2} \{k_h\tilde{\mu}_\alpha\}\,.$$

Hence,

$$\frac{\varepsilon_h}{\Delta_{\alpha h}} \sim \tilde{\mu}_\alpha \frac{\{h\bar{\mu}_j\}}{\{k_h\tilde{\mu}_\alpha\}} \qquad \forall \alpha \neq j$$

and, again by (54), we see that (50) is satisfied. This proves the "only if" part of the statement. Concerning the "if" part, assume that (50) holds, and notice that together with the previous formula it yields (54) for $\alpha \neq i, j$. Moreover, in particular $\{h\bar{\mu}_j\} \to 0$ which, due to (55), establishes the validity of (54) also for $\alpha = j$. Looking now at $\alpha = i$ observe that

$$k_h\tilde{\mu}_i = \lfloor h\bar{\mu}_j \rfloor\, \tilde{\mu}_i = h - \tilde{\mu}_i\, \{h\bar{\mu}_j\}$$

eventually yields $\{k_h\tilde{\mu}_i\} = 1 - \tilde{\mu}_i\, \{h\bar{\mu}_j\}$. This implies that $\Delta_{ih} \sim \mu_i/h^2$, and then

$$\frac{\varepsilon_h}{\Delta_{ih}} \sim \tilde{\mu}_i\, \{h\bar{\mu}_j\} \to 0,$$

proving the validity of (54) also for $\alpha = i$. ∎

Some comments about condition (50) are in order. Looking more closely at the denominator in (50), notice that (51) yields

$$k_n \frac{\mu_\alpha}{\mu_j} = h_n \frac{\mu_\alpha}{\mu_i} - \frac{\mu_\alpha}{\mu_j} \left\{ h_n \frac{\mu_j}{\mu_i} \right\} = \left\{ h_n \frac{\mu_\alpha}{\mu_i} \right\} + \left\lfloor h_n \frac{\mu_\alpha}{\mu_i} \right\rfloor - \frac{\mu_\alpha}{\mu_j} \left\{ h_n \frac{\mu_j}{\mu_i} \right\},$$

which in turn implies the congruence

$$\left\{ k_n \frac{\mu_\alpha}{\mu_j} \right\} = \left\{ h_n \frac{\mu_\alpha}{\mu_i} \right\} - \frac{\mu_\alpha}{\mu_j} \left\{ h_n \frac{\mu_j}{\mu_i} \right\} \mod (1) \tag{56}$$

for every $\alpha = 1, \dots, N$. In order to compute the fraction in (50), we have to deduce from (56) some *exact* equality on the fractional parts. This is done in next lemma for a particular case, and it leads to a useful sufficient condition for (50).

Lemma 7.3 *Assume that* (Q) *holds and that for some* $\alpha \neq i, j$

$$\lim_{n \to \infty} \frac{\left\{ h_n \frac{\mu_j}{\mu_i} \right\}}{\left\{ h_n \frac{\mu_\alpha}{\mu_i} \right\}} = 0 \tag{57}$$

along a diverging integer sequence h_n*. Then condition (50) holds for the same* α*.*

Proof. Because of (57), the corresponding right hand side in (56) rewrites as

$$\{h_n \bar{\mu}_\alpha\} - \tilde{\mu}_\alpha \{h_n \bar{\mu}_j\} = \{h_n \bar{\mu}_\alpha\} (1 - \varepsilon_n)$$

where $0 < \varepsilon_n \to 0$ (the bar and the tilde have the same meaning as in the previous lemma). Hence, for large n, this quantity lies in $(0, 1)$, so that the equality in (56) holds without the mod (1) limitation. This yields $\{k_n \tilde{\mu}_\alpha\} \sim \{h_n \bar{\mu}_\alpha\}$, proving that condition (50) is satisfied for the considered α. ■

The previous arguments suggest that the classical Kronecker Theorem on Diophantine approximation is the main tool when looking for asymptotic resonances. Hereafter we present a slightly refined version of this theorem, in a form suited for our purposes.

Theorem 7.4 (Kronecker) *Assume that the real numbers* $1, \alpha_1, \dots, \alpha_M$ *are linearly independent over* $\mathbf{Z}$*, and that* P *is a given natural number. Then, for every* $c_1, \dots, c_M \in [0, 1]$ *and every* $p_0, p_1, \dots, p_M \in \mathbf{N}$ *there exists an integer sequence* h_n *such that*

$$\lim_{n \to \infty} \{h_n \alpha_j\} = c_j \qquad \forall j,$$

$$h_n = p_0 \mod (P) \qquad \forall n,$$

$$\lfloor h_n \alpha_j \rfloor = p_j \mod (P) \quad \forall n, \ \forall j.$$

The conclusion about the fractional parts is the standard one (Theorems 442–443 of [8]). Notice that by a diagonal process we may also prescribe that in each limit the convergence holds from above (or from below). We will use this further freedom in the proof of the next proposition. By similar arguments, for every j, we may also decide at which rate we want c_j to be approximated by $\{h_n\alpha_j\}$; this will be used in the proof of Proposition 7.6.
The new part in the statement is the prescription on the integer parts and it will be used in Proposition 7.7. As it will be clear after the proof, the above theorem is in fact equivalent to the classical one.

Proof. We will prove the (new part of the) statement for $0 \le c_j < 1$ only, since by a standard diagonal process one can recover also the case $c_j = 1$.
Assume now that $h_n = p_0 + P\ell_n$ for suitable integers ℓ_n, and then use the classical Kronecker Theorem to choose ℓ_n such that

$$\lim_{n\to\infty} \{\ell_n\alpha_j\} = \left\{\frac{c_j + p_j - p_0\alpha_j}{P}\right\}^+ \qquad \forall j.$$

Notice that for every j,

$$h_n\alpha_j = (p_0 + P\ell_n)\alpha_j = c_j + P\left(\{\ell_n\alpha_j\} - \left\{\frac{c_j + p_j - p_0\alpha_j}{P}\right\}\right) + p_j \quad \mod (P),$$

so that the choice of ℓ_n yields

$$\begin{aligned} \{h_n\alpha_j\} &= c_j + P\left(\{\ell_n\alpha_j\} - \left\{\frac{c_j + p_j - p_0\alpha_j}{P}\right\}\right) \to c_j, \\ \lfloor h_n\alpha_j\rfloor &= p_j \quad \mod (P), \end{aligned}$$

which prove the claim. ■

Going back to asymptotic resonances, in the next proposition we deal with the particular case in which the frequencies are independent over the integers as a whole, not only pairwise as required by (Q). Although this covers a generic choice of frequencies, we will see in the next section that this is not always the case for the Fermi–Pasta–Ulam model.

Proposition 7.5 *Assume that $\mu_1, \ldots, \mu_N$ are linearly independent over* $\mathbf{Z}$*. Then any μ_i is asymptotically resonant with any other μ_j with $j \neq i$.*

Proof. Using the same notation as in the proof of Lemma 7.2, we see that the numbers $\bar\mu_1, \ldots, \bar\mu_N$ are also independent over $\mathbf{Z}$. Since by construction $\bar\mu_i = 1$, the Kronecker Theorem allows us to choose an integer sequence h_n such that, for instance,

$$\{h_n\bar\mu_j\} \to 0 \qquad \text{and} \qquad \{h_n\bar\mu_\alpha\} \to 1/2 \quad \forall \alpha \neq i, j$$

as $n \to +\infty$, the second condition to be considered only when $N \ge 3$. To conclude we apply Lemma 7.2, via Lemma 7.3. ■

When the frequencies are dependent over **Z**, we cannot expect that the same result is true. Consider for instance a case where (Q) holds, preventing dependence on pairs, but

$$\mu_\alpha = p\mu_i + \mu_j \tag{58}$$

for a given choice of indexes i, j, α and some (nonzero) $p \in \mathbf{Z}$. The equality

$$\left\{ h_n \frac{\mu_\alpha}{\mu_i} \right\} = \left\{ h_n \frac{\mu_j}{\mu_i} \right\}$$

holds for every choice of the integer h_n. Looking now at the k_n defined by (51), notice that the right hand side of (56) takes the form

$$\left\{ h_n \frac{\mu_\alpha}{\mu_i} \right\} - \frac{\mu_\alpha}{\mu_j} \left\{ h_n \frac{\mu_j}{\mu_i} \right\} = \left(1 - \frac{\mu_\alpha}{\mu_j} \right) \left\{ h_n \frac{\mu_j}{\mu_i} \right\} = -p \, \frac{\mu_i}{\mu_j} \left\{ h_n \frac{\mu_j}{\mu_i} \right\}.$$

Choose the integer sequence h_n such that (53) is satisfied. If $p < 0$, we have

$$\left\{ k_n \frac{\mu_\alpha}{\mu_j} \right\} = -p \, \frac{\mu_i}{\mu_j} \left\{ h_n \frac{\mu_j}{\mu_i} \right\},$$

which prevents condition (50) to be satisfied, so that μ_i cannot be right asymptotically resonant with μ_j.

The next proposition says that the presence of *ternary relations* over **Z** such as (58), namely relations involving *exactly three frequencies*, are the only obstruction to asymptotic resonance.

Proposition 7.6 *Assume that (Q) holds, and let $i \neq j$ be fixed. Suppose that no ternary relation over* **Z** *involves μ_i and μ_j at the same time. Then μ_i is asymptotically resonant with μ_j.*

Proof. Denote by d the dimension of the **Q**–vector space generated by the **Q**–relations among the frequencies $\mu_1, \ldots, \mu_N$. Then d of them, say μ_α where $\alpha \in A$, may be expressed as a function of the remaining $N - d$, say μ_β where $\beta \in B$. Namely,

$$\mu_\alpha = \sum_{\beta \in B} c_{\alpha\beta} \, \mu_\beta \qquad \forall \alpha \in A, \tag{59}$$

for some suitable $c_{\alpha\beta} \in \mathbf{Q}$. As a consequence of (59), the frequencies μ_β's must be linearly independent over **Q**. The idea is now to act on the μ_β's with the Kronecker Theorem, to obtain some control on the μ_α's.

First of all we notice that, because of (Q), it is not restrictive to assume that $i, j \in B$. Indeed, assume for instance that $i \in A$ and $j \in B$; in the relation corresponding to $\alpha = i$ in (59), because of $\mu_i/\mu_j \notin \mathbf{Q}$, it must be $c_{il} \neq 0$ for some

$l \in B \setminus \{j\}$. Hence with a trivial manipulation one can take away μ_l from all the right hand sides, the final effect being to swap the position of l and i. Similar arguments apply starting from $i, j \in A$.
Recalling that there are no ternary relations involving μ_i and μ_j, we also see that the set B cannot reduce to $\{i, j\}$; more precisely, for every $\alpha \in A$ the set

$$B_\alpha = \{\beta \in B \setminus \{i, j\} \mid c_{\alpha\beta} \neq 0\}$$

must be nonempty. We denote by β_α its smallest element.
Rewriting (59) using integer coefficients, namely as

$$a_\alpha \mu_\alpha = \sum_{\beta \in B} b_{\alpha\beta}\ \mu_\beta \qquad \forall \alpha \in A, \tag{60}$$

with $a_\alpha, b_{\alpha\beta} \in \mathbf{Z}$, we can assume without loss of generality that, for every $\alpha \in A$, $a_\alpha > 0$ and the integer numbers in the set $\{a_\alpha\} \cup \{b_{\alpha\beta} \mid \beta \in B\}$ have no common divisors.
As a consequence of (60), if h_n is any integer sequence, then

$$a_\alpha \{h_n\ \bar{\mu}_\alpha\} = \sum_{\beta \in B \setminus \{i\}} b_{\alpha\beta} \{h_n\ \bar{\mu}_\beta\} + \sum_{\beta \in B} b_{\alpha\beta} \lfloor h_n\ \bar{\mu}_\beta \rfloor \quad \mod (a_\alpha)$$

for every $\alpha \in A$; here the bar denotes, as usual, the division by μ_i.
Choose now the sequence h_n such that

$$\{h_n\ \bar{\mu}_j\} \sim \frac{1}{n^{j+N}} \tag{61}$$

and

$$\{h_n\ \bar{\mu}_\beta\} \sim \frac{1}{n^\beta} \qquad \forall \beta \in B \setminus \{i, j\} \tag{62}$$

as $n \to +\infty$. This is possible due to the Kronecker Theorem, since the $\bar{\mu}_\beta$'s are $\mathbf{Q}$–independent and $\bar{\mu}_i = 1$.
Notice that condition (52) is trivially satisfied in $B \setminus \{i, j\}$, while condition (57) (and hence (50)) is satisfied because the fractional part in (61) vanishes faster than all the ones in (62).
To conclude the proof we have to show that the same happens to every μ_α with $\alpha \in A$.
To this aim notice that, by definition of β_α, we have $b_{\alpha\beta_\alpha} \neq 0$ and

$$\sum_{\beta \in B \setminus \{i\}} b_{\alpha\beta} \{h_n\ \bar{\mu}_\beta\} \sim \frac{b_{\alpha\beta_\alpha}}{n^{\beta_\alpha}}.$$

Moreover, up to subsequences, there is an integer m_α with $0 \leq m_\alpha < a_\alpha$, such that

$$\sum_{\beta \in B} b_{\alpha\beta} \lfloor h_n\ \bar{\mu}_\beta \rfloor = m_\alpha \qquad \mod (a_\alpha)$$

for every $\alpha \in A$ and every n. If $m_\alpha > 0$ then

$$\{h_n\, \bar{\mu}_\alpha\} = \frac{1}{a_\alpha} \left(m_\alpha + \frac{b_{\alpha\beta_\alpha}}{n^{\beta_\alpha}}(1 + o(1)) \right),$$

so that $\{h_n\, \bar{\mu}_\alpha\} \to m_\alpha / a_\alpha > 0$. The previous relation takes place also when $m_\alpha = 0$ and $b_{\alpha\beta_\alpha} > 0$, proving that in this case

$$\{h_n\, \bar{\mu}_\alpha\} = \frac{b_{\alpha\beta_\alpha}}{a_\alpha\, n^{\beta_\alpha}}(1 + o(1)).$$

Finally, if $m_\alpha = 0$ but $b_{\alpha\beta_\alpha} < 0$, we have

$$\{h_n\, \bar{\mu}_\alpha\} = 1 + \frac{b_{\alpha\beta_\alpha}}{a_\alpha\, n^{\beta_\alpha}}(1 + o(1)) \to 1.$$

In all these cases, conditions (57) and (52) follow immediately. ∎

Next we discuss the effect of ternary relations. As an example we notice that although relation (58) is "bad" for $p < 0$ (it prevents right asymptotic resonance), a change in the coefficient of μ_α may remove the obstruction to the asymptotic resonance of μ_i with μ_j, for all p. For instance, it can be proved that the ternary relation $2\mu_\alpha = p\mu_i + \mu_j$, with a nonzero $p \in \mathbf{Z}$, is always harmless, for every notion of asymptotic resonance.
To settle the problem in general we begin by noticing that, because of (Q), given any three different frequencies, up to multiplication by integers there is at most one ternary relation involving them.
Assume then that

$$\begin{cases} a_\alpha \mu_\alpha = b_\alpha \mu_i + c_\alpha \mu_j \\ \\ a_\alpha > 0,\ (a_\alpha, b_\alpha, c_\alpha) = 1 \end{cases} \qquad \alpha \in \Gamma \tag{63}$$

is the complete list of the ternary relations with integer coefficients involving μ_i and μ_j at the same time, normalized to rule out repetitions. Notice that all the frequencies μ_α in the left hand side must be distinct.
The next proposition is the main result of this section; it provides a simple criterion for asymptotic resonance.

Proposition 7.7 *Assume that (Q) holds, and that (63) lists all the normalized ternary relations involving μ_i and μ_j. Then μ_i is right asymptotically resonant with μ_j if and only if there exist integer numbers p, q such that*

$$b_\alpha p + c_\alpha q \neq 0 \mod (a_\alpha) \tag{64}$$

for every $\alpha \in \Gamma$ for which

$$b_\alpha < 0 \qquad \textit{and} \qquad c_\alpha > 0. \tag{65}$$

For left asymptotic resonance the same condition is to be considered for the $\alpha \in \Gamma$ for which

$$c_\alpha < 0, \tag{66}$$

while for asymptotic resonance the α's to be considered are those which satisfy either (65) or (66).

In case no relation in (63) satisfies (65) or (66), the corresponding conclusions are automatically true.

Proof. We prove only the statement concerning right asymptotic resonance, since due to the form of the denominator in (50), it is the most delicate. The other cases follow similarly. To prove the "if" part, we modify the proof of the previous proposition, maintaining the same notation. Consider then the list of relations (60) and the associated partition A, B of the set $\{1, \ldots, N\}$. Recall that this is a very special base for the relations, since it is adapted to the choice $i, j \in B$.
We claim that, as a consequence, all the relations listed in (63) must appear in the list (60). Fix indeed a ternary relation in (63), corresponding to some $\alpha_0 \in \Gamma$. Since $i, j \in B$ and the μ_β's with $\beta \in B$ are independent over the integers, we must have $\alpha_0 \in A$. Now if the relation in (60) corresponding to α_0 does not coincide with the one fixed in (63), then a suitable linear combination of the two yields a nontrivial relation among the μ_β's, which is impossible.
Choose now an integer sequence h_n according to (61) and (62).
We have already proved that this choice rules out the relations in (60) corresponding to the α's lying $B \setminus \{i, j\}$ or in $A \setminus \Gamma$. It remains to look at Γ, which we partition into the three subsets

$$\begin{aligned} \Gamma_1 &= \{\alpha \in \Gamma \mid c_\alpha < 0\}, \\ \Gamma_2 &= \{\alpha \in \Gamma \mid c_\alpha > 0,\ b_\alpha < 0\}, \\ \Gamma_3 &= \{\alpha \in \Gamma \mid c_\alpha > 0,\ b_\alpha > 0\}. \end{aligned}$$

First we show that also Γ_1 is ruled out by (53), which holds due to choice of h_n satisfying (61). Indeed, by now standard computations, for every $\alpha \in \Gamma$ we have

$$\begin{aligned} a_\alpha \{h_n\, \bar{\mu}_\alpha\} &= c_\alpha \{h_n\, \bar{\mu}_j\} + b_\alpha h_n + c_\alpha \lfloor h_n\, \bar{\mu}_j \rfloor & \mod (a_\alpha) \\ &= c_\alpha \{h_n\, \bar{\mu}_j\} + m_{n\alpha} & \mod (a_\alpha) \end{aligned}$$

for a unique integer satisfying $0 < m_{n\alpha} \leq a_\alpha$. As usual, the bar denotes the division by μ_i. Due to (61), when $\alpha \in \Gamma_1$ this yields (for large n) the exact equality

$$\{h_n\, \bar{\mu}_\alpha\} = \frac{1}{a_\alpha} \left(m_{n\alpha} + c_\alpha \{h_n\, \bar{\mu}_j\}\right).$$

As a consequence,

$$\liminf_{n\to+\infty} \{h_n\, \bar{\mu}_\alpha\} \geq \frac{1}{a_\alpha} > 0,$$

which proves condition (57) and then the claim.
We now use condition (64) to rule out the relations corresponding to $\Gamma_2 \cup \Gamma_3$. Define $P = \prod_{\alpha\in\Gamma_2\cup\Gamma_3} a_\alpha$ and assume that, in addition to (61) and (62), the sequence h_n also satisfies

$$h_n = p \bmod (P) \qquad \text{and} \qquad \lfloor h_n\, \bar{\mu}_j \rfloor = q \bmod (P), \tag{67}$$

where the integers p and q are exactly the ones considered in (64). This is always possible, due to Theorem 7.4.
With standard manipulations, we see that

$$a_\alpha \{h_n\, \bar{\mu}_\alpha\} = c_\alpha \{h_n\, \bar{\mu}_j\} + b_\alpha p + c_\alpha q \qquad \bmod (a_\alpha)$$

holds for every $\alpha \in \Gamma_2 \cup \Gamma_3$. Here we used the fact that any equality modulo P is also true modulo a_α for every $\alpha \in \Gamma_2 \cup \Gamma_3$. Denote by m_α the unique integer satisfying

$$\begin{cases} 0 \leq m_\alpha < a_\alpha \\ m_\alpha = b_\alpha p + c_\alpha q \quad \bmod (a_\alpha). \end{cases}$$

Since $c_\alpha > 0$ and (61) holds, the exact equality

$$\{h_n\, \bar{\mu}_\alpha\} = \frac{1}{a_\alpha} \left(m_\alpha + c_\alpha \{h_n\, \bar{\mu}_j\}\right)$$

must be true for large values of n.
If now $m_\alpha > 0$ for some α, then we obtain $\{h_n\, \bar{\mu}_\alpha\} \to m_\alpha/a_\alpha > 0$, which rules out the corresponding relations by means of (57).
Because of (64) and (65), this is certainly the case at least for every $\alpha \in \Gamma_2$. Thus it remains to consider only the $\alpha \in \Gamma_3$ for which $m_\alpha = 0$, in which case

$$\{h_n\, \bar{\mu}_\alpha\} = \frac{c_\alpha}{a_\alpha}\; \{h_n\, \bar{\mu}_j\}\,. \tag{68}$$

Condition (57) is not satisfied, and we have to look at the most general condition (50). The right hand side of (56) is

$$\{h_n\bar{\mu}_\alpha\} - \tilde{\mu}_\alpha \{h_n\bar{\mu}_j\} = \frac{c_\alpha\mu_j - a_\alpha\mu_\alpha}{a_\alpha\mu_j}\; \{h_n\bar{\mu}_j\} \bmod (1) = -\tilde{\mu}_i\, \frac{b_\alpha}{a_\alpha}\; \{h_n\bar{\mu}_j\} \bmod (1),$$

where the tilde denotes the division by μ_j.
Since $b_\alpha > 0$ due to $\alpha \in \Gamma_3$, from (56) we obtain that the equality

$$\{k_n\tilde{\mu}_\alpha\} = 1 - \tilde{\mu}_i \frac{b_\alpha}{a_\alpha}\; \{h_n\bar{\mu}_j\}$$

must hold for large n. Hence $\{k_n\tilde{\mu}_\alpha\} \to 1$, so that condition (50) is satisfied.
We turn to the "only if" part of the statement. Let h_n be an integer sequence satisfying (53). Up to subsequences we may certainly assume that

$$h_n = p_* \bmod (P) \qquad \text{and} \qquad \lfloor h_n\, \bar{\mu}_j \rfloor = q_* \bmod (P)$$

hold for some suitable integers p_* and q_*, where, as above, $P = \prod_{\alpha\in\Gamma_2\cup\Gamma_3} a_\alpha$. Assume now that (64) is false, and choose $\alpha \in \Gamma_2$ such that $b_\alpha p_* + c_\alpha q_* = 0$ $\bmod(a_\alpha)$. Since

$$a_\alpha \{h_n\, \bar{\mu}_\alpha\} = c_\alpha\ \{h_n\, \bar{\mu}_j\} + b_\alpha h_n + \lfloor h_n\, \bar{\mu}_j \rfloor \qquad \bmod (a_\alpha)$$

and $c_\alpha > 0$, we deduce that the equality (68) holds also in this case. Repeating the same arguments as above, with $b_\alpha < 0$, leads now to the equality

$$\{k_n\tilde{\mu}_\alpha\} = -\tilde{\mu}_i\, \frac{b_\alpha}{a_\alpha}\ \{h_n\bar{\mu}_j\}\,,$$

which prevents condition (50) to be satisfied. Hence μ_i cannot be asymptotically resonant with μ_j, and the proof is complete. ■

8 Asymptotic resonance in the FPU problem

In this section we specialize the general results on asymptotic resonance to the case of the Fermi–Pasta–Ulam model with N particles and fixed ends considered in the Introduction. In this case

$$Ax\cdot x = \sum_{j=0}^{N}(x_{j+1} - x_j)^2$$

(where $x_0 = x_{N+1} = 0$) and it is well known (see for instance [18]) that the eigenvalues μ_j^2 of A and the corresponding eigenvectors e_j satisfy

$$\mu_j = 2\sin\left(\frac{j\pi}{2(N+1)}\right) \quad \text{and} \quad (e_j)_k = \sin\left(\frac{kj\pi}{(N+1)}\right) \tag{69}$$

for $j, k = 1, \ldots, N$. In this section we are interested in the μ_j's only, and in particular in the properties of the $\mathbf{Q}$–vector space of their $\mathbf{Q}$–relations, which we denote by $\mathcal{M}_N$.
The next lemma characterizes the very favorable situation where μ_j's are rationally independent as a whole (namely $\mathcal{M}_N$ is trivial); though we are really interested in ternary relations only, we report it for sake of completeness. It is due to Hemmer ([9]) and it is well known in connection to the Fermi–Pasta–Ulam problem. Since the proof is unpublished, for the convenience of the reader we present here a slightly simplified version.

Lemma 8.1 *$\mathcal{M}_N$ is trivial if and only if $N+1$ is a power of 2 or a prime; otherwise its dimension is $N - \varphi(2(N+1))$, where φ is the Euler function.*

We recall that the Euler function is defined on the natural numbers by

$$\varphi(n) \;=\; \#\{1 \le a \le n \mid (a,n) = 1\}.$$

Notice that $\varphi(1) = 1$ and $\varphi(n) < n$ for all $n > 1$. If n is prime, $\varphi(n) = n - 1$; otherwise inequality $\varphi(n) < n-1$ holds. The practical computation of φ depends on the well known properties

$$\begin{cases} \varphi(ab) = \varphi(a)\,\varphi(b) & \text{if } \ (a,b) = 1 \\ \varphi(p^{h+1}) = p^h(p-1) & \text{if } \ p \ \text{ is prime}, \quad h \ge 0, \end{cases} \tag{70}$$

where a, b and h are all naturals (see [8], Chapter V).
The other object we will use in the proof is $\Phi_n(x)$, the cyclotomic polynomial of order n. We recall that, by definition, it is the element of minimal degree in $\mathbf{Q}[x]$ having $e^{2\pi i/n}$ as a root, normalized by taking equal to one the leading coefficient. It is well known (see Theorem 4.10 in [14]) that it can be represented as

$$\Phi_n(x) = \prod_{(a,n)=1} (x - e^{2\pi i a/n}), \tag{71}$$

so that its degree is exactly $\varphi(n)$.

Proof. Let

$$\sum_{j=1}^{N} a_j \mu_j = 0$$

be a nontrivial element of $\mathcal{M}_N$, and denote by m be the smallest index such that $a_j = 0$ when $j > m$. If we set $\zeta = \exp(2\pi i/4(N+1))$, then μ_j is (the double of) the imaginary part of ζ^j. Hence, defining

$$G(x) = \sum_{j=1}^{m} a_j (x^{m+j} - x^{m-j}) \,,$$

we can rewrite our relation in the more convenient polynomial form $G(\zeta) = 0$. Of course $G \in \mathbf{Q}[x]$ and, since $a_m \ne 0$ by definition of m, its degree is $\partial G = 2m$. Moreover, the manifest symmetry of G yields the formula

$$x^{\partial G} G(1/x) = -G(x).$$

Since $G(\zeta) = 0$, the polynomial $G(x)$ must be divisible by $\Phi_{4(N+1)}(x)$. Write then $G(x) = f(x)\Phi_{4(N+1)}(x)$ for some $f \in \mathbf{Q}[x]$. Using the properties of G, it is not difficult to check that

$$\partial f = 2m - 2\,\varphi(2(N+1)), \tag{72}$$

$$x^{\partial f} f(1/x) = -f(x). \tag{73}$$

Formula (72) depends on the fact that $\varphi(4(N+1)) = 2\varphi(2(N+1))$, which follows from the rules (70) after decomposing $N+1$ into its odd and even parts. Concerning (73), use the representation (71) to prove that $\Phi_{4(N+1)}(x)$ has the same symmetry as G but for the sign.

The key point is now to realize that necessarily $\partial f \geq 2$; indeed, f cannot be constant because of (73), and it has even degree due to (72). We conclude that $N \geq m > \varphi(2(N+1))$. In case

$$N \ \leq \ \varphi(2(N+1)), \tag{74}$$

we obtain a contradiction, and therefore $\mathcal{M}_N$ must be trivial.

On the other hand, if $N > \varphi(2(N+1))$, then for every $r = 1, \ldots, N - \varphi(2(N+1))$ we consider the polynomial $f \in \mathbf{Q}[x]$ defined by

$$f(x) = x^{2r} - 1,$$

which has even degree and satisfies (73). Then, by simply reversing the previous construction, we obtain a non trivial element in $\mathcal{M}_N$. Since moreover every polynomial $f(x)$ of even degree and satisfying (73) uniquely decomposes as

$$f(x) = \sum_{j=1}^{\partial f/2} c_j \, x^{\partial f/2 - j}(x^{2j} - 1)$$

for suitable $c_j \in \mathbf{Q}$, the relations obtained in this way are clearly a base for $\mathcal{M}_N$. This proves the claim concerning the dimension of $\mathcal{M}_N$.

To conclude the proof, we show that (74) holds if and only if $N+1$ is a prime or a power of 2. To this aim, begin by writing

$$N + 1 = 2^h D$$

with h and D integers and D odd; of course, if $h = 0$ then necessarily $D > 1$. Using the rules (70), condition (74) rewrites as

$$D - \frac{1}{2^h} \ \leq \ \varphi(D). \tag{75}$$

If $h > 0$ this condition holds if and only if $\varphi(D) \geq D$, which is equivalent to $D = 1$; this is the case where $N+1$ is a power of 2.

On the other hand, if $h = 0$ then (75) rewrites as $\varphi(D) \geq D - 1$ which, since $D > 1$, is equivalent to say that $D = N + 1$ is prime. ■

Although Hemmer's result covers completely some particular cases of importance, and even provides an algorithm to construct a base for $\mathcal{M}_N$, its use is not really advisable to handle the general case, mainly because it can be very hard (especially for large N) to isolate ternary relations from a base for $\mathcal{M}_N$.

An efficient alternative is provided by a result of Conway and Jones (see also [12] and [13]). In [4] indeed the authors set up an algorithmic way to solve the classical problem of the vanishing sums of roots of unity, and make the explicit computations for low order sums. As a byproduct of Theorem 7 in [4], it turns out that

$$\begin{cases} \sin(\pi/3+\theta) - \sin(\pi/3-\theta) - \sin(\theta) = 0, & 0 < \theta < \pi/6, \\ \sin(3\pi/10) - \sin(\pi/10) - \sin(\pi/6) = 0 \end{cases} \tag{76}$$

is the complete list (up to multiplication by rationals) of the **Q**–relations involving *up to three* sines, in which the all the angles lie in $(0, \pi/2)$ and are moreover rational with π.

The frequencies $\mu_1, \ldots, \mu_N$ considered in (69) fit into this framework, for every dimension N. As a first relevant consequence, there are no relations involving two frequencies only, so that *condition* (Q) *holds for every* N.

Moreover, the result of Conway and Jones permits the complete classification of the ternary relations among frequencies, for every N. Straightforward computations show that the μ_j's may satisfy the first relation in (76) only if 3 divides $N{+}1$, and the second one when also 5 does. This yields the following proposition.

Proposition 8.2 *There are no ternary relations among the frequencies in (69), unless* $3 \mid (N{+}1)$*. In this case moreover, if* $5 \nmid (N{+}1)$ *then all the ternary relations are described (up to multiplication by rationals) by*

$$\mu_{\frac{2}{3}(N+1)+\gamma} - \mu_{\frac{2}{3}(N+1)-\gamma} - \mu_\gamma = 0, \tag{77}$$

where the integer γ *ranges over* $1, \ldots, (N{+}1)/3 - 1$*. Finally, if also* $5 \,|(N{+}1)$*, then the special relation*

$$\mu_{\frac{3}{5}(N+1)} - \mu_{\frac{1}{5}(N+1)} - \mu_{\frac{1}{3}(N+1)} = 0 \tag{78}$$

must be added to the previous list.

We are finally ready to describe all the asymptotic resonances among the frequencies μ_j's, merging the above characterization together with the results proved in the last section.

For every $N \in \mathbf{N}^+$ and $i \in \{1, \ldots, N\}$ define the sets

$$\Gamma_R(N,i) = \begin{cases} \{i, \frac{2}{3}(N{+}1) + i\} & \text{if } \ 1 \le i < \frac{1}{3}(N{+}1), \ i \ne \frac{1}{5}(N{+}1) \\ \{i, \frac{2}{3}(N{+}1) + i, \frac{3}{5}(N{+}1)\} & \text{if } \ i = \frac{1}{5}(N{+}1) \\ \{i, \frac{3}{5}(N{+}1)\} & \text{if } \ i = \frac{1}{3}(N{+}1) \\ \{i, \frac{4}{3}(N{+}1) - i\} & \text{if } \ \frac{1}{3}(N{+}1) < i \le \frac{2}{3}(N{+}1) \\ \{i\} & \text{if } \ \frac{2}{3}(N{+}1) < i \le N \end{cases}$$

and

$$\Gamma_L(N,i) = \begin{cases} \{i\} & \text{if } \ 1 \le i \le \frac{2}{3}(N{+}1), i \ne \frac{3}{5}(N{+}1) \\ \{i, \frac{1}{5}(N{+}1), \frac{1}{3}(N{+}1)\} & \text{if } \ i = \frac{3}{5}(N{+}1) \\ \{i, \frac{4}{3}(N{+}1) - i, i - \frac{2}{3}(N{+}1)\} & \text{if } \ \frac{2}{3}(N{+}1) < i \le N. \end{cases}$$

Theorem 8.3 *Let $\mu_1, \ldots, \mu_N$ be the frequencies of the N–particle Fermi–Pasta–Ulam problem with fixed ends. If $N+1$ is not a multiple of 3, then μ_i is asymptotically resonant with μ_j for every $j \neq i$. If $N+1$ is a multiple of 3, then*
a) μ_i is right asymptotically resonant with μ_j if and only if $j \notin \Gamma_R(N,i)$.
b) μ_i is left asymptotically resonant with μ_j if and only if $j \notin \Gamma_L(N,i)$.
c) μ_i is asymptotically resonant with μ_j if and only if $j \notin \Gamma_R(N,i) \cup \Gamma_L(N,i)$.

Notice that, when $3 \,|\, (N+1)$, some of the forbidden values for j are integer only when also $5 \,|\, (N+1)$: to exclude them also when $5 \nmid (N+1)$ is just a trick to decrease the number of different cases to be considered.

Proof. When 3 does not divide $N{+}1$, the result follows from Propositions 7.6 and 8.2. Let us then assume that 3 divides $N+1$ and apply Proposition 7.7. Also in this case we only prove the part concerning right asymptotic resonance. Notice that in every ternary relation the coefficients are unitary. Hence condition (64) cannot be satisfied, and we have an obstruction to the asymptotic resonance each time there are ternary relations of the type (63) satisfying (65). Now, the list (63) depends of course on the choice of the indexes $i \neq j$. To prove the statement we have then to distinguish many different cases, each time deciding whether (65) is fulfilled or not.

Assume first that

$$1 \le i < \frac{1}{3}(N+1),$$

and notice that we can fit one of the ternary relations (77) only if $\gamma = i$. If we moreover set $j = 2(N+1)/3 + i$, then (65) is satisfied, preventing the asymptotic resonance of μ_i with μ_j. On the contrary, (65) is not verified if we set $j = 2(N+1)/3 - i$. This explains the first row in the definition of $\Gamma_R(N,i)$.
When also 5 divides $N+1$, then (78) comes into play, the only possible choice for i being $i = (N+1)/5$. To explain the second row in $\Gamma_R(N,i)$, we use the very same arguments as above to show that $j = 3(N+1)/5$ produces an obstruction, while $j = (N+1)/3$ does not. All the remaining cases can be worked out similarly, so that they are left to the reader. ∎

9 Nonlinear coupling in the FPU problem

In this section we investigate the validity of the nonlinear coupling condition used in Theorem 4.6 in the case of the FPU β–model.
Without loss of generality we take $\beta = 1$, so that the nonlinear potential is

$$W(x) = \frac{1}{4}\left\{x_1^4 + (x_2 - x_1)^4 + \ldots + (x_N - x_{N-1})^4 + x_N^4\right\}.$$

We also recall that in the case of the FPU model, the matrix A admits N eigenvalues μ_j^2, with corresponding eigenvectors e_j given by (69).
Our aim is to describe for which N, i, j, the number

$$W_{ij} = 6\mu_i^2 W(e_j) - \mu_j^2 W''(e_j) \cdot e_i^2$$

is negative. In this case the nonlinear coupling condition between μ_j and μ_i is satisfied.
To begin with we notice that if we set, in agreement with (69), $(e_j)_0 = 0$ and $(e_j)_{N+1} = 0$, then we can write more conveniently

$$W(e_j) = \frac{1}{4}\sum_{k=0}^{N}\left((e_j)_{k+1} - (e_j)_k\right)^4$$

and

$$W''(e_j) \cdot e_i^2 = 3\sum_{k=0}^{N}\left((e_j)_{k+1} - (e_j)_k\right)^2\left((e_i)_{k+1} - (e_i)_k\right)^2.$$

The computation of the numbers W_{ij} requires only elementary algebraic manipulations. We begin with the following lemma, which will be used repeatedly; in its statement, δ denotes the Kronecker delta. We omit the proof, since it can be easily obtained in many ways.

Lemma 9.1 *Let $q \in \mathbf{Z}$ be such that $-(N-1) \le q \le 2(N+1)$ and define*

$$S_N(q) = \sum_{k=0}^{N} \cos\Big((2k+1)\frac{q\pi}{N+1}\Big). \tag{79}$$

Then $S_N(q) = (N+1)(\delta_{q,0} - \delta_{q,N+1})$.

In particular, the cases we will use, with $1 \le i, j \le N$, are
$S_N(j) = 0$,
$S_N(2j) = -(N+1)\delta_{2j,N+1}$,
$S_N(j+i) = -(N+1)\delta_{j+i,N+1}$,
$S_N(j-i) = (N+1)\delta_{j-i,0} = (N+1)\delta_{j,i}$.
We now turn to the computation of $W(e_j)$ and $W''(e_j) \cdot e_i^2$.

Lemma 9.2 *We have*

$$W(e_j) = \frac{N+1}{32}\mu_j^4(3 - \delta_{2j,N+1}). \tag{80}$$

Proof. By an elementary trigonometric identity and the definition of μ_j, we can write

$$(e_j)_{k+1} - (e_j)_k = \sin\Big((k+1)\frac{j\pi}{N+1}\Big) - \sin\Big(k\frac{j\pi}{N+1}\Big) = \mu_j \cos\Big((k+\frac{1}{2})\frac{j\pi}{N+1}\Big).$$

Therefore

$$W(e_j) = \frac{1}{4}\mu_j^4 \sum_{k=0}^{N} \cos^4\left((k+\frac{1}{2})\frac{j\pi}{N+1}\right).$$

Now, $\cos^4\theta = \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$, so that with the help of the previous corollary,

$$\begin{aligned} W(e_j) &= \frac{1}{4}\mu_j^4 \sum_{k=0}^{N} \Big(\frac{3}{8} + \frac{1}{2}\cos\Big((2k+1)\frac{j\pi}{N+1}\Big) + \frac{1}{8}\cos\Big((2k+1)\frac{2j\pi}{N+1}\Big)\Big) \\ &= \frac{1}{4}\mu_j^4(\frac{3}{8}(N+1)) + \frac{1}{2}S_N(j) + \frac{1}{8}S_N(2j)) = \frac{N+1}{32}\mu_j^4(3 - \delta_{2j,N+1}). \end{aligned}$$

■

Lemma 9.3 *We have*

$$W''(e_j) \cdot e_i^2 = \frac{3(N+1)}{8}\mu_i^2\mu_j^2(2 + \delta_{j,i} - \delta_{j+i,N+1}). \tag{81}$$

Proof. By the same argument used at the beginning of the proof of the previous lemma, it is easy to see that

$$W''(e_j)\cdot e_i^2 = 3\mu_i^2\mu_j^2 \sum_{k=0}^{N} \cos^2\Big((k+\frac{1}{2})\frac{j\pi}{N+1}\Big)\cos^2\Big((k+\frac{1}{2})\frac{i\pi}{N+1}\Big).$$

Applying the identity $2\cos^2\theta = 1+\cos(2\theta)$ we obtain, with the same arguments as in Lemma 9.2

$$W''(e_j)\cdot e_i^2 = \frac{3}{4}\mu_i^2\mu_j^2 \sum_{k=0}^{N}(1+\cos\big((2k+1)\tfrac{j\pi}{N+1}\big))(1+\cos\big((2k+1)\tfrac{i\pi}{N+1}\big))$$

$$= \frac{3}{4}\mu_i^2\mu_j^2\Big(N+1+S_N(j)+S_N(i)+\sum_{k=0}^{N}\cos\big((2k+1)\tfrac{j\pi}{N+1}\big)\cos\big((2k+1)\tfrac{i\pi}{N+1}\big)\Big)$$

$$= \frac{3}{4}\mu_i^2\mu_j^2\Big(N+1+\frac{1}{2}\sum_{k=0}^{N}\Big(\cos\big((2k+1)\tfrac{(j+i)\pi}{N+1}\big)+\cos\big((2k+1)\tfrac{(j-i)\pi}{N+1}\big)\Big)\Big)$$

$$= \frac{3(N+1)}{8}\mu_i^2\mu_j^2(2+\delta_{j,i}-\delta_{j+i,N+1}).$$

■

We are now ready to describe when the nonlinear coupling condition is satisfied.

Proposition 9.4 *Let $W_{ij} = 6\mu_i^2 W(e_j) - \mu_j^2 W''(e_j)\cdot e_i^2$. Then*

$$W_{ij} < 0 \quad \text{if and only if} \quad \begin{cases} j+i \neq N+1 & \text{if } N \text{ is even} \\ j+i\neq N+1 \text{ or } i=j=\frac{N+1}{2} & \text{if } N \text{ is odd.}\end{cases} \tag{82}$$

Proof. By (80) and (81) we immediately see that

$$W_{ij} = -\frac{3(N+1)}{16}\mu_i^2\mu_j^4(1+\delta_{2j,N+1}+2\delta_{j,i}-2\delta_{j+i,N+1}).$$

Then, if $j+i\neq N+1$, W_{ij} is negative, whatever the parity of N. If $j+i=N+1$ and $i\neq j$, then also $2j\neq N+1$, and W_{ij} is positive. The only possibility left is $j+i=N+1$ and $i=j$; in this case of course $2j=N+1$, so that W_{ij} is again negative. This last case may take place only if N is odd. ■

In Theorem 4.6 the nonlinear coupling condition $W_{ij}<0$ is used in combination with the requirement that μ_i be right asymptotically resonant with μ_j. Since as we have pointed out in Section 4 in this case necessarily $j\neq i$, the previous proposition yields the following simple result.

Corollary 9.5 *Assume that μ_i is right asymptotically resonant with μ_j. Then*

$$W_{ij} < 0 \quad \text{if and only if} \quad j+i\neq N+1. \tag{83}$$

The preponderance of negative terms explains, we believe, the numerical outcomes reported in [1].
As a final result, we report the specialization of Corollary 1.4 to the Fermi–Pasta–Ulam case.

Corollary 9.6 *For every $N \geq 2$, the N–particle Fermi–Pasta–Ulam problem with fixed ends admits a sequence of periodic solutions whose $\mathcal{C}^2$ norms tend to zero and whose minimal periods tend to infinity.*

Proof. We just treat $N \geq 3$. By Theorem 8.3, it is immediate to check that $2 \notin \Gamma_R(N, 1)$ for every N; therefore μ_1 is right asymptotically resonant with μ_2. Since $1 + 2 = 3 < N + 1$, the preceding corollary assures that $W_{12} < 0$. The application of Corollary 1.4 completes the proof. ∎

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